القيم الشبكية للبنيات المزدوجه المتناسقة

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الملخص:

فى هذا البحث تم تقديم مفهوم البنيات المزدوجه المتناسقة ذات القيم الشبكية . وأيضا تمت دراسة العلاقة الطبيعية بين البنيات ال مزدوجه المتناسقة والفضاءات التوبولوجية المزدوجه والبنيات التقاربية المنتظمة المزدوجه ودراسة خواص هذه التركيبات.

Journal of the Association of Arab Universities for Basic and Applied Sciences (2011) **10**, 33–41

University of Bahrain
Journal of the Association of Arab Universities for Basic and Applied Sciences
www.elsevier.com/locate/jaaubas
www.sciencedirect.com

ORIGINAL ARTICLE

Lattice valued double syntopogenous structures

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Available online 2 August 2011

KEYWORDS
Double fuzzy topological spaces;
Topogenous structure;
Quasi-proximity

Abstract In this paper, we introduce the concept of lattice valued double fuzzy syntopogenous structures in framework of double fuzzy topology (proximity and uniformity). Some fundamental properties of them are established. Finally, a natural links between double fuzzy syntopogenous structure, double fuzzy topology, double fuzzy proximity and double fuzzy uniformity are given.

1. Introduction and preliminaries

Kubiak (1985) and Šostak (1985) introduced the notion of (L-) fuzzy topological space as a generalization of L-topological spaces (originally called (L-)fuzzy topological spaces by Chang (1968) and Goguen (1973). It is the grade of openness of an L-fuzzy set. A general approach to the study of topological-type structures on fuzzy powersets was developed in Höhle (1980), Höhle and Šostak (1995), and Kubiak (1985).

As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassov (1986). Recently, Çoker (1997) and Çoker et al., 1996 introduced the notion of intuitionistic fuzzy topological space using intuitionistic fuzzy sets. Samanta and Mondal (2002) introduced the notion of intuitionistic gradation of openness which a generalization of both of L-fuzzy topological spaces and the topology of intuitionistic fuzzy sets (Çoker, 1997).

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Peer review under responsibility of University of Bahrain.
doi:10.1016/j.jaubas.2011.06.007

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Working under the name "intuitionistic" did not continue because doubts were thrown about the suitability of this term, especially when working in the case of complete lattice L. These doubts were quickly ended in 2005 by Garcia and Rodabaugh, 2005. They proved that this term is unsuitable in mathematics and applications. They concluded that they work under the name "double".

Csaszar (1963) gave a new method for the foundation of general topology based on the theory of syntopogenous structure to develop a unified approach to the three main structures of set-theoretic topology: topologies, uniformities and proximities. This enabled him to evolve a theory including the foundations of the three classical theories of topological spaces, uniform spaces and proximity spaces. In the case of the fuzzy structures there are at least two notions of fuzzy syntopogenous structures, the first notion worked out in (Katsaras, 1988, 1985a, 1983) presents a unified approach to the theories of Chang (1968) fuzzy topological spaces, Hutton fuzzy uniform spaces (Hutton, 1977), Katsaras fuzzy proximity spaces (Katsaras, 1985b, 1980, 1979) and Artico fuzzy proximity (Artico and Moresco, 1984). The second notion worked out in Katsaras (1991, 1990) agree very well with Lowen fuzzy topological spaces (Lowen, 1976), Lowen–Hohle fuzzy uniform spaces (Lowen, 1981) and Artico–Moresco fuzzy proximity spaces (Artico and Moresco, 1984).

In this paper, we establish the concept of double fuzzy syntopogenous structures as a unified approach to theores of (Hohle and Rodabaugh) double fuzzy topology, double fuzzy proximity spaces and double fuzzy uniformity spaces. Some fundamental properties of them are established. Finally, the relationship among double fuzzy syntopogenous structures, double fuzzy topology, double fuzzy proximity and double fuzzy uniformity is studied.

Throughout this paper, let X be a nonempty set and $L = (L, \leqslant, \vee, \wedge, \perp, \top)$ a completely distributive lattice where $\perp(\top)$ denotes the universal lower (upper) bound.

Definition 1.1. CQML, the category of **complete quasi-monoidal lattices**, (Rodabaugh, 2003).

Comprises the following data, where composition and identities are taken from **SET:**

(1) **Objects**: $(L, \leqslant, \odot)$ where $\odot : L \times L \to L$ is isotone and $\top \odot \top = \top$.
(2) **Morphisms:** All **SET** morphisms preserves $\odot$, $\top$ and arbitrary $\vee$.

Definition 1.2. Categories related to **CQML** (Rodabaugh, 2003).

(1) **QUML**, the category of quasi-uniform monoidal lattices is the full subcategory of **CQML** for which $\leftarrow \odot \leftarrow$ is associative, commutative and $\leftarrow \top \leftarrow$ is identity.
(2) **DQML**, the category of deMorgan quasi-monoidal lattices is the full subcategory of **CQML** for which * is an order-reversing involution and each morphism preserves the involution.
(3) **QUANT**, the category of **quantales** is the full subcategory of **CQML** for which $\odot$ is distributive over arbitrary joins, i.e.,

$$(\bigvee_{i\in\Gamma} r_i) \odot s = \bigvee_{i\in\Gamma}(r_i \odot s).$$

(4) **CQUANT**, the category of **coquantales** is the full subcategory of **CQML** for which $\odot$ is distributive over arbitrary meets, i.e.,

$$(\bigwedge_{i\in\Gamma} r_i) \odot s = \bigwedge_{i\in\Gamma}(r_i \odot s).$$

(5) **DQUAT,** the category of **deMorgan**, **quasi-uniform monoidal quantales.**
In this paper, for each $(L, \leqslant, \odot, *) \in$ **DQUAT**, we define $x \oplus y = (x^* \odot y^*)^*$.
(6) **DBIQUAT = DQUAT $\cap$ T COQUANT.**
(7) **CMVAL**, the category of complete MV-algebra is the full subcategory of **DBIQUAT** for which it satisfies

(MV) $(x \mapsto y) \mapsto y = x \vee y$, for all x, $y \in L$ where $x \mapsto y$ is defined by $x \mapsto y = \vee\{z| x \odot z \leqslant y\}$ and $x^* = x \mapsto \perp$.

Definition 1.3 (Kim and Ko, 2008). Let $(L, \leqslant, \odot, \oplus, *) \in |$ **DQUAT**$|$ and $\phi : X \to Y$ be a function. For each $x, \{y, z \in L, y_i | i \in \Gamma, f, g \in L^X$ and $f_i \in L^Y$. we have:

(1) If $y \leqslant z, (x \odot y) \leqslant (x \odot z)$ and $(x \oplus y) \leqslant (x \odot z)$.
(2) $x \odot y \leqslant x \wedge y \leqslant x \vee y \leqslant x \oplus y$.
(3) $\bigwedge_{i\in\Gamma} y_i^* = (\bigvee_{i\in\Gamma} y_i)^*$ and $\bigvee_{i\in\Gamma} y_i^* = (\bigwedge_{i\in\Gamma} y_i)^*$.
(4) $x \oplus (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \oplus y_i)$.

Definition 1.4 (Cetkin and Aygun, 2010). The maps $\mathcal{T}, \mathcal{T}^* : L^X \to L$ is called double fuzzy topology on X if it satisfies the following conditions:

$(LO1)$ $\mathcal{T}(f) \leqslant (\mathcal{T}^*(f))^*$, for all $f \in L^X$,
$(LO2)$ $\mathcal{T}(1_\emptyset) = \mathcal{T}(1_X) = \top$ and $\mathcal{T}^*(1_\emptyset) = \mathcal{T}^*(1_X) = \perp$,
$(LO3)$ $\mathcal{T}(f_1 \odot f_2) \geqslant \mathcal{T}(f_1) \odot \mathcal{T}(f_1)$ and $\mathcal{T}^*(f_1 \odot f_2) \leqslant \mathcal{T}^*(f_1) \oplus \mathcal{T}^*(f_1)$, for each f_1, $f_2 \in L^X$,
$(LO4)$ $\mathcal{T}(\bigvee_{i\in\Delta} f_i) \geqslant \bigwedge_{i\in\Delta} \mathcal{T}(f_i)$ and $\mathcal{T}^*(\bigvee_{i\in\Delta} f_i) \leqslant \bigvee_{i\in\Delta} \mathcal{T}^*(f_i)$ for each $f_i \in L^X$, $i \in \Delta$.

The pair $(X, \mathcal{T}, \mathcal{T}^*)$ is called an double fuzzy topological space (dfts, for short).

Let $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ and $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ be dfts's. A map $\phi : X \to Y$ is called fuzzy continuons iff $\mathcal{T}_2 \leqslant \mathcal{T}_1 \circ \phi_L^{\leftarrow}$ and $\mathcal{T}_2^* \geqslant \mathcal{T}_1^* \circ \phi_L^{\leftarrow}$.

2. Double fuzzy topogenous order and double fuzzy topologies

Definition 2.1. A maps η, $\eta^* : L^X \times L^X \to L$ is called double fuzzy semi-topogenous order on X if it satisfies the following axioms:

$(LST1)$ $\eta(f, g) \leqslant (\eta^*(f, g))^*$, for all f, $g \in L^X$,
$(LST2)$ $\eta(1_X, 1_X) = \eta(1_\emptyset, 1_\emptyset) = \top$ and $\eta^*(1_X, 1_X) = \eta^*(1_\emptyset, 1_\emptyset) = \perp$,
$(LST3)$ If$\eta(f, g) \neq \perp$ and $\eta^*(f, g) \neq \top$, then $f \leqslant g$,
$(LST4)$ If $f_1 \leqslant f$, $g_1 \leqslant g$, then $\eta(f_1, g_1) \leqslant \eta(f, g)$ and $\eta^*(f_1, g_1) \geqslant \eta^*(f, g)$.

Proposition 2.2. *Let (η, η^*) be a double fuzzy semi-topogenous order on X and let the mappings η^s, $\eta^{s^*} : L^X \times L^X \to L$ defined by $\eta^s(f,g) = \eta(g^*, f^*)\psi$ and $\eta^{s^*}(f,g) = \eta^*(g^*, f^*)$, $\forall f$, $g \in L^X$. Then (η^s, η^{s^*}) is double fuzzy semi-topogenous order on X.*

Definition 2.3. A double fuzzy semi-topogenous order (η, η^*) is called symmetric if $(\eta, \eta^*) = (\eta^s, \eta^{s^*})$, that is,

(LST4) $\eta(f,g) = \eta(g^*, f^*)$ and $\eta^*(f, g) = \eta^*(g^*, f^*)$, $\forall f$, $g \in L^X$.

Definition 2.4. A double fuzzy semi-topogenous order (η, η^*) is called double fuzzy topogenous if for any f, f_1, f_2, g, g_1, $g_2 \in L^X$.

(LST5) $\eta(f_1 \oplus f_2, g) \geqslant \eta(f_1, g) \odot \eta(f_2, g)$ and $\eta^*(f_1 \oplus f_2, g) \leqslant \eta^*(f_1, g) \oplus \eta^*(f_2, g)$,
(LST6) $\eta(f, g_1 \odot g_2) \geqslant \eta(f, g_1) \odot \eta(f, g_2)$ and $\eta^*(f, g_1 \odot g_2) \leqslant \eta^*(f, g_1) \oplus \eta^*(f, g_2)$.

Definition 2.5. A double fuzzy semi-topogenous order (η, η^*) is called perfect if (LST7) $\eta(\bigvee_{i\in\Gamma} f_i, g) \geqslant \bigwedge_{i\in\Gamma} \eta(f_i, g)$ and $\eta^*(\bigvee_{i\in\Gamma} f_i, g) \leqslant \bigvee_{i\in\Gamma} \eta^*(f_i, g)$, for any $\{g, f_i : i \in \Gamma\} \subset L^X$.

An perfect double fuzzy semi-topogenous order (η, η^*) is called biperfect if (LST8) $\eta(f, \bigwedge_{i\in\Gamma} g_i) \geqslant \bigwedge_{i\in\Gamma} \eta(f, g_i)$ and $\eta^*(f, \bigwedge_{i\in\Gamma} g_i) \leqslant \bigvee_{i\in\Gamma} \eta^*(f, g_i)$, for any $\{g, f_i : i \in \Gamma\} \subset L^X$.

Theorem 2.6. *Let (η_1, η_1^*) and (η_2, η_2^*) be perfect (respectively, double fuzzy topogenous, biperfect) double fuzzy semi-topogenous order on X. Define the compositions $\eta_1 \circ \eta_2$ of η_1 and η_2 and $\eta_1^* \circ \eta_2^*$ of η_1^* and η_2^* on X by $\eta_1 \circ \eta_2(f, g) = \bigvee_{h\in L^X}[\eta_1(f, h) \odot \eta_2(h, g)]$ and $\eta_1^* \circ \eta_2^*(f, g) = \bigwedge_{h\in L^X}[\eta_1^*(f, h) \oplus \eta_2^*(h, g)]$.*

Then $(\eta_1 \circ \eta_2, \eta_1^* \circ \eta_2^*)$*is perfect (resp. double fuzzy topogenous, biperfect) double fuzzy semi- topogenous order on X.*

Proof. Let (η_1, η_1^*) and (η_2, η_2^*) be perfect double fuzzy semi-topogenous order on X. Then (LST3) If $\eta_1 \circ \eta_2(f,g) \neq \perp$ and $\eta_1^* \circ \eta_2^*(f,g) \neq \top$ then there exists $h \in L^X$ such that $\eta_1 \circ \eta_2(f,g) \geqslant \eta_1(f,h) \odot \eta_2(h,g) \neq \perp$ and $\eta_1^* \circ \eta_2^*(f,g) \leqslant \eta_1^*(f,h) \oplus \eta_2^*(h,g) \neq \top$. It implies $f \leqslant h \leqslant g$.

(LST7) It is proved from:

$$\begin{aligned}\eta_1 \circ \eta_2(\bigvee_{i\in\Gamma} f_i, g) &= \bigvee_{h\in L^X} \left[\eta_1(\bigvee_{i\in\Gamma} f_i, h) \odot \eta_2(h,g)\right]\\ &\geqslant \bigwedge_{i\in\Gamma} \left[\bigvee_{h\in L^X} [\eta_1(f_i,h) \odot \eta_2(h,g)]\right]\\ &= \bigwedge_{i\in\Gamma} \eta_1 \circ \eta_2(f_i, g),\end{aligned}$$

$$\begin{aligned}\eta_1^* \circ \eta_2^*(\bigvee_{i\in\Gamma} f_i, g) &= \bigwedge_{h\in L^X} \left[\eta_1^*(\bigvee_{i\in\Gamma} f_i, h) \oplus \eta_2^*(h,g)\right]\\ &\leqslant \bigvee_{i\in\Gamma} \left[\bigwedge_{h\in L^X} [\eta_1^*(f_i,h) \oplus \eta_1^*(h,g)]\right]\\ &= \bigvee_{i\in\Gamma} \eta_1^* \circ \eta_2^*(f_i, g).\end{aligned}$$

Others are easily proved. □

Definition 2.7. A double fuzzy syntopogenous structure on $X\Psi$ is a non-empty family $\Upsilon_X\Psi$ of double fuzzy topogenous orders on X satisfying the following two conditions:

(LS1) $\Upsilon_X\Psi$ is directed, i.e. given two double fuzzy topogenous orders (η_1, η_1^*), $(\eta_2, \eta_2^*) \in \Upsilon_X$, there exists a double fuzzy topogenous order $(\eta_1, \eta^*) \in \Upsilon_X$ such that $\eta \geqslant \eta_1,\ \eta_2$ and $\eta^* \leqslant \eta_1^*,\ \eta_2^*$.

(LS2) For every $(\eta, \eta^*) \in \Upsilon_X$, there exists$(\eta_1^{\cdot}\eta_1^*) \in \Upsilon_X$, such that $\eta \leqslant \eta_1 \circ \eta_2$and $\eta^* \geqslant \eta_1^* \circ \eta_2^*$.

Definition 2.8. A double fuzzy syntopogenous structure $\Upsilon_X\Psi$ is called double fuzzy topogenous if $\Upsilon_X\Psi$ consists of a single element. In this case, $\Upsilon_X\Psi = \{(\eta, \eta^*)\}$ is called a double fuzzy topogenous structure, denoted by $\Upsilon_X\Psi = \{(\eta, \eta^*)\} = (\eta, \eta^*)$ and (X, Υ_X) is called double fuzzy topogenous space.

A double fuzzy syntopogenous structure $\Upsilon_X\Psi$ is called perfect (resp. biperfect, symmetric) if each double fuzzy topogenous order$(\eta, \eta^*) \in \Upsilon_X$ is perfect (resp.respect, biperfect, symmetric).

Theorem 2.9. *Let* (η, η^*) *topogenous order on X. The mapping* $C_\eta : L^X \times L_0 \times L_1 \to L^X$*, is defined by*

$$C_{\eta,\mu^*}(f, r, s) = \wedge\{g^* \in L^X : \eta(g, f^*) > r \quad \textit{and}\ \eta^*(g, f^*) \leqslant s\}.$$

For each $f, f_1, f_2 \in L^X$ *and* $r, r_1, r_2 \in L_0$, $s, s_1, s_2 \in L_1$, *we have the following properties:*

(1) $C_{\eta,\eta^*}(1_\emptyset, r, s) = 1_\emptyset$.
(2) $f \leqslant C_{\eta,\eta^*}(f, r, s)$.
(3) *If* $f_1 \leqslant f_2$, *then* $C_{\eta,\eta^*}(f_1, r, s) \leqslant C_{\eta,\eta^*}(f_2, r, s)$.
(4) $C_{\eta,\eta^*}(f_1 \oplus f_2, r \odot r_1, s \oplus s_1) \leqslant C_{\eta,\eta^*}(f_1, r, s) \oplus C_{\eta,\eta^*}(f_2, r_1, s_1)$.
(5) *If* $r_1 \leqslant r_2$ *and* $s_1 \geqslant s_2$, *then* $C_{\eta,\eta^*}(f, r_1, s_1) \leqslant C_{\eta,\eta^*}(f, r_2, s_2)$.
(6) *If* (X, η, η^*) *is double topogenous space, then*$C_{\eta,\eta^*}(C_{\eta,\eta^*}(f, r, s), r, s) \leqslant C_{\eta,\eta^*}(f, r, s)$.

Proof. (1) Since $\eta(1_\emptyset, 1_\emptyset) = \top$ and $\eta^*(1_\emptyset, 1_\emptyset) = \perp$ $C_{\eta,\eta^*}(1_\emptyset, r, s) = 1_\emptyset$.

(2) Since $\eta(g, f^*) \neq \perp$ and $\eta^*(g, f^*) \neq \top$, $g \leqslant f^*$ implies $f \leqslant C_{\eta,\eta^*}(f, r, s)$.

(3) and (5) are easily proved.

(4) Suppose that there exists $f_1, f_2 \in L^X\psi$ such that

$$C_{\eta,\eta^*}(f_1 \oplus f_2, r \odot r_1, s \oplus s_1)\imath\ C_{\eta,\eta^*}(f_1, r, s) \oplus C_{\eta,\eta^*}(f_2, r_1, s_1).$$

By the definition of C_{η,η^*} there exists $g_1,\ g_2 \in L^X$ with $\eta(g_1, f_1^*)\imath r$, $\eta^*(g_1, f_1^*) \leqslant s$, $\eta(g_2, f_2^*)\imath r_1$ and $\eta^*(g_2, f_2^*) \leqslant s_1$ such that $C_{\eta,\eta^*}(f_1 \oplus f_2, r \odot r_1, s \oplus s_1)\imath g_1^* \oplus g_2^*$.

On the other hand,

$$\begin{aligned}{}_\Psi\eta(g_1 \odot g_2, (f_1 \oplus f_2)^*) &\geqslant \eta(g_1 \odot g_2, f_1^*) \odot \eta(g_1 \odot g_2, f_2^*) \quad (\text{by LST6})\\ &\geqslant \eta(g_1, f_1^*) \odot \eta(g_2, f_2^*) \qquad (\text{by LST4})\\ \eta r \odot r_1,&\\ \eta^*(g_1 \odot g_2, (f_1 \oplus f_2)^*) &\leqslant \eta^*(g_1 \oplus g_2, f_1^*) \oplus \eta^*(g_1 \odot g_2, f_2^*) \quad (\text{by LST6})\\ &\leqslant \eta^*(g_1, f_1^*) \oplus \eta^*(g_2, f_2^*) \qquad (\text{by LST4})\\ &\leqslant s \oplus s_1.\end{aligned}$$

It implies $C_{\eta,\eta^*}(f_1 \oplus f_2, r \odot r_1, s \oplus s_1) \leqslant (g_1 \odot g_2)^* = g_1^* \oplus g_2^*\Psi$. It is a contradiction.

(6) Let $\eta(g, f^*) \nless r$ and $\eta^*(g, f^*) \leqslant s$. Then $g^* \geqslant C_{\eta,\eta^*}(f, r, s)$. Since (X, η, η^*) is double fuzzy topogenous space, by (LS2) of Definition 2.7, there exists (η, η^*) such that $\eta \leqslant \eta \circ \eta$ and $\eta^* \geqslant \eta^* \circ \eta^*$. It follows

$$\eta(g, f^*) \leqslant \eta \circ \eta(g, f^*) \quad \text{and} \quad \eta^*(g, f^*) \geqslant \eta^* \circ \eta^*(g, f^*).$$

Since $\eta \circ \eta(g, f^*) \nless r$ and $\eta^* \circ \eta^*(g, f^*) \nless s$, there exists $h \in L^X$ such that
$\eta \circ \eta(g, f^*) \geqslant \eta(g, h) \odot \eta(h, f^*) \nless r$ and $\eta^* \circ \eta^*(g, f^*) \leqslant \eta^*(g, h) \oplus \eta^*(h, f^*) \leqslant s$.

Hence, $g^* \geqslant C_{\eta,\eta^*}(h^*, r, s)$ and $h^* \geqslant C_{\eta,\eta^*}(f, r, s)$. Thus, $g^* \geqslant C_{\eta,\eta^*}(C_{\eta,\eta^*}(f, r, s), r, s)$. So,

$$C_{\eta,\eta^*}(C_{\eta,\eta^*}(f, r, s), r, s) \leqslant C_{\eta,\eta^*}(f, r, s).$$ □

Theorem 2.10. *Let* (X, η, η^*) *be a double fuzzy topogenous order space. Define the maps* $\mathcal{T}_\eta, \mathcal{T}^*_{\eta^*} : L^X \to L$ *by*

$$\leftarrow\leftarrow \mathcal{T}_\eta(f) = \vee\{r \in L_0 \mid C\eta(f^*, r, s) \leqslant f^*\},$$
$$\mathcal{T}^*_{\eta^*}(f) = \wedge\{s \in L_1 \mid C\eta(f^*, r, s) \leqslant f^*\}.$$

Then, $(\mathcal{T}_\eta, \mathcal{T}^*_{\eta^*})$ is double fuzzy topology on X induced by (η, η^*).

Proof. (LO1) clear.

(LO2) $C_{\eta,\eta^*}(1_\emptyset, r, s) = 1_\emptyset$ and $C_{\eta,\eta^*}(1_X, r, s) = 1_X$ for all $r \in L_0, s \in L_1$, $\mathcal{T}_\eta(1_\emptyset) = \mathcal{T}_\eta(1_X) = \top$ and $\mathcal{T}^*_{\eta^*}(1_\emptyset) = \mathcal{T}^*_{\eta^*}(1_X) = \perp$.

(LO3) Suppose there exist $f_1, f_2 \in L^X$such that

$\mathcal{T}_{\eta}(f_1 \odot f_2) \not\geqslant \mathcal{T}_{\eta}(f_1) \odot \mathcal{T}_{\eta}(f_2)$ and $\mathcal{T}^*_{\eta^*}(f_1 \odot f_2) \not\leqslant \mathcal{T}^*_{\eta^*}(f_1) \oplus \mathcal{T}^*_{\eta^*}(f_2)$.

By the definition of $(\mathcal{T}_{\eta}, \mathcal{T}^*_{\eta^*})$ there exists $r_i \in L_0$, $s_i \in L_1$ with $f_i^* \geqslant C_{\eta,\eta^*}(f_i^*, r_i, s_i)$, $i = 1, 2$ such that

$\mathcal{T}_{\eta}(f_1 \odot f_2) \not\geqslant r_1 \odot r_2$ and $\mathcal{T}^*_{\eta^*}(f_1 \odot f_2) \not\leqslant s_1 \oplus s_2$.

Put $r = r_1 \odot r_2$ and $s = s_1 \oplus s_2$. By (4–5) of Theorem 2.9, we have

$C_{\eta,\eta^*}((f_1 \odot f_2)^*, r, s) \leqslant (f_1 \odot f_2)^*$.

Consequently, $\mathcal{T}_{\eta}(f_1 \odot f_2) \geqslant r$ and $\mathcal{T}^*_{\eta^*}(f_1 \odot f_2) \leqslant s$. Hence (LO3) holds.

(LO4) Suppose there exists a family $\{f_j \in L^X | j \in \Gamma\}$ such that

$\mathcal{T}_{\eta}(\bigvee_{i\in\Gamma} f_i) \not\geqslant \bigwedge_{i\in\Gamma} \mathcal{T}_{\eta}(f_i)$ or $\mathcal{T}^*_{\eta^*}(\bigvee_{i\in\Gamma} f_i) \not\leqslant \bigvee_{i\in\Gamma} \mathcal{T}^*_{\eta^*}(f_i)$.

For each $j \in \Gamma$, there exists $r_j \in L_0, s_j \in L_1$ *with* $f_j^* \geqslant C_{\eta,\eta^*}(f_j^*, r_j, s_j)$ such that

$\mathcal{T}_{\eta}(\bigvee_{j\in\Gamma} f_j) \not\geqslant \bigwedge_{j\in\Gamma} r_j$ or $\mathcal{T}^*_{\eta^*}(\bigvee_{j\in\Gamma} f_j) \not\leqslant \bigvee_{j\in\Gamma} s_j$.

Put $r = \bigwedge_{j\in\Gamma} r_j$ and $s = \bigvee_{j\in\Gamma} s_j$. By (4–5) of Theorem 2.9, we have $C_{\eta,\eta^*}((\bigvee_{j\in\Gamma} f_j)^*, r, s) \leqslant (\bigvee_{j\in\Gamma} f_j)^*$.

Consequently, $\mathcal{T}_{\eta}(\bigvee_{j\in\Gamma} f_j) \geqslant r$ and $\mathcal{T}^*_{\eta^*}(\bigvee_{j\in\Gamma} f_j) \leqslant s$. Hence (LO4) holds. □

Definition 2.11.

Let (X, η_1, η_1^*) and (Y, η_2, η_2^*) be double fuzzy topogenous order spaces. A function $\Psi\phi : (X, \eta_1, \eta_1^*) \to (Y, \eta_2, \eta_2^*)$ is said to be topogenous continuous if $\Psi\eta_2(f, g) \leqslant \eta_1(\phi_L^{\leftarrow}(f), \phi_L^{\leftarrow}(g))$ and $\eta_2^*(f, g) \geqslant \eta_1^*(\phi_L^{\leftarrow}(f), \phi_L^{\leftarrow}(g))$, Ψ *for each* $f, g \in L^Y$.

Theorem 2.12. Let (X, η_1, η_1^*), (Y, η_2, η_2^*) and (Z, η_3, η_3^*) be double fuzzy topogenous order spaces. If $\Phi : (X, \eta_1, \eta_1^*) \to (Y, \eta_2, \eta_2^*)$ and $\Psi : (Y, \eta_2, \eta_2^*) \to (Z, \eta_3, \eta_3^*)$ are topogenous continuous, then $\Psi \circ \Phi : (X, \eta_1, \eta_1^*) \to (Z, \eta_3, \eta_3^*)$ is topogenous continuous.

Proof.

It follows that, for each $f, g \in L^Z$

$$\begin{aligned}
\eta_1((\psi\circ\phi)_L^{\leftarrow}(f), (\psi\circ\phi)_L^{\leftarrow}(g)) &= \eta_1\big(\phi_L^{\leftarrow}(\psi_L^{\leftarrow}(f)), \phi_L^{\leftarrow}\big(\psi_L^{\leftarrow}(g)\big)\big) \\
&\geqslant \eta_2\big(\psi_L^{\leftarrow}(f), \psi_L^{\leftarrow}(g)\big) \\
&\geqslant \eta_3(f, g), \\
\eta_1^*((\psi\circ\phi)_L^{\leftarrow}(f), (\psi\circ\phi)_L^{\leftarrow}(g)) &= \eta_1^*\big(\phi_L^{\leftarrow}\big(\psi_L^{\leftarrow}(f)\big), \phi_L^{\leftarrow}\big(\psi_L^{\leftarrow}(g)\big)\big) \\
&\leqslant \eta_2^*(\psi_L^{\leftarrow}(f), \psi_L^{\leftarrow}(g)) \\
&\leqslant \eta_3^*(f, g). \quad \square
\end{aligned}$$

Theorem 2.13. *Let (X, η_1, η_1^*) and (Y, η_2, η_2^*) be double fuzzy topogenous order spaces. If $\Phi : (X, \eta_1, \eta_1^*) \to (Y, \eta_2, \eta_2^*)$ Ψ is topogenous continuous, then it satisfies the following statements:*

(1) $\phi_L^{\rightarrow}(C_{\eta_1,\eta_1^*}(f, r, s)) \leqslant C_{\eta_2,\eta_2^*}(\phi_L^{\rightarrow}(f), r, s)$, *for each* $f \in L^X$, $r \in L_0$, $s \in L_1$,
(2) $C_{\eta_1,\eta_1^*}(\phi_L^{\leftarrow}(g), r, s) \leqslant \phi_L^{\leftarrow}(C_{\eta_2,\eta_2^*}(g, r, s))$ *for each* $g \in L^X$, $r \in L_0$, $s \in L_1$,
(3) $\phi : (X, \mathcal{T}_{\eta_1}, \mathcal{T}^*_{\eta_1^*}) \to (Y, \mathcal{T}_{\eta_2}, \mathcal{T}^*_{\eta_2^*})$ *is fuzzy continuous.*

Proof. (1) Suppose there exist $f \in L^X$, $r \in L_0$, $s \in L_1$ such that

$\phi_L^{\rightarrow}(C_{\eta_1,\eta_1^*}(f, r, s)) \cdot C_{\eta_2,\eta_2^*}(\phi_L^{\rightarrow}(f), r, s)$.

By the definition of C_{η_2,η_2^*} there exists $g \in L^Y$ with $\eta_2(g, (\phi_L^{\rightarrow}(f))^*)r$ and $\eta_2^*(g, (\phi_L^{\rightarrow}(f))^*) \leqslant s$ such that
$\Psi\phi_L^{\rightarrow}(C_{\eta_1,\eta_1^*}(f, r, s)) \not\leqslant g^*$. (A)

By the topogenous continuity of ϕ we have,

$\eta_1(\phi_{\mathrm{L}}^{\leftarrow}(g), \phi_{\mathrm{L}}^{\leftarrow}((\phi^{\rightarrow}{}_{\mathrm{L}}(f))^*) \geqslant \eta_2(g, (\phi_{\mathrm{L}}^{\rightarrow}(f))^*)\eta r$,
$\eta_1^*(\phi_{\mathrm{L}}^{\leftarrow}(g), \phi_{\mathrm{L}}^{\leftarrow}((\phi_{\mathrm{L}}^{\rightarrow}(f))^*) \leqslant \eta_2^*(g, (\phi_{\mathrm{L}}^{\rightarrow}(f))^*) \leqslant s$.

Since $\eta_1(\phi_{\mathrm{L}}^{\leftarrow}(g), f^*) \geqslant \eta_1(\phi_{\mathrm{L}}^{\leftarrow} g, \phi_{\mathrm{L}}^{\leftarrow}((\phi_{\mathrm{L}}^{\rightarrow}(f))^*)$ and $\eta_1^*(\phi_{\mathrm{L}}^{\leftarrow}(g), f^*) \leqslant \eta_1^*(\phi_{\mathrm{L}}^{\leftarrow}(g), \phi_{\mathrm{L}}^{\leftarrow}((\phi_{\mathrm{L}}^{\rightarrow}(f))^*)$ we have $C_{\eta_1,\eta_1^*}(f, r, s) = (\phi_L^{\leftarrow}(g))^* = \phi_L^{\leftarrow}(g^*)$. Thus $\phi_L^{\rightarrow}(C_{\eta_1,\eta_1^*}(f, r, s)) \leqslant g^*$. It is a contradiction for equation (A).

(2) For each $g \in L^Y$, $r \in L_0$ and $s \in L_1$, put $f = \phi_L^{\leftarrow}(g)$. From (1),

$$\begin{aligned}
\phi_L^{\rightarrow}(C_{\eta_1,\eta_1^*}(\phi_L^{\leftarrow}(g), r, s)) &\leqslant C_{\eta_2,\eta_2^*}(\phi_L^{\rightarrow}(\phi_L^{\leftarrow}(g)), r, s) \\
&\leqslant C_{\eta_2,\eta_2^*}(g, r, s).
\end{aligned}$$

It implies

$$\begin{aligned}
C_{\eta_1,\eta_1^*}(\phi_L^{\leftarrow}(g), r, s) &\leqslant \phi_L^{\leftarrow}(\phi_L^{\rightarrow}(C_{\eta_1,\eta_1^*}(\phi_L^{\leftarrow}(g), r, s))) \\
&\leqslant \phi_L^{\leftarrow}(C_{\eta_2,\eta_2^*}(g, r, s)).
\end{aligned}$$

(3) From (2), $C_{\eta_2,\eta_2^*}(g, r, s) = g$ implies $C_{\eta_1,\eta_1^*}(\phi_L^{\leftarrow}(g), r, s) = \phi_L^{\leftarrow}(g)$. It is easily proved from Theorem 2.10. □

3. Double fuzzy quasi-proximities

Definition 3.1 (Cetkin and Aygun, 2010). A maps $\delta, \delta^* : L^X \times L^X \to L$ is called a double fuzzy quasi-proximity on X if it satisfies the following axioms:

(LP1) $\delta(f, g) \leqslant (\delta^*(f, g))^*, \forall f, g \in L^X$.
(LP2) $\delta(1_X, 1_\emptyset) = \delta(1_\emptyset, 1_X) = \perp$ and $\delta^*(1_X, 1_\emptyset) = \delta^*(1_\emptyset, 1_X) = \top$. (LP3) If $\delta(f, g) \neq \top$ and $\delta^*(f, g) \neq \perp$ then $f \leqslant g^*$. item(LO4) If $f \leqslant g$ then $\delta(f, h) \leqslant \delta(g, h)$ and $\delta^*(f, h) \geqslant \delta^*(g, h)$,
(LP5) $\delta(f_1 \odot f_2, g_1 \oplus g_2) \leqslant \delta(f_1, g_1) \oplus \delta(f_2, g_2)$ and $\delta^*(f_1 \odot f_2, g_1 \oplus g_2) \geqslant \delta^*(f_1, g_1) \odot \delta^*(f_2, g_2)$.
(LP6) For any $f, g \in L^X$, there exists $h \in L^X$ such that $\delta(f, g) \geqslant \bigwedge_{h\in L^X} \{\delta(f, h) \oplus \delta(h^*, g)\}$ and $\delta^*(f, g) \leqslant \bigvee_{h\in L^X} \{\delta^*(f, h) \odot \delta^*(h^*, g)\}$. The triple (X, δ, δ^*) is double fuzzy quasi-proximity space.

A double fuzzy quasi-proximity space (X, δ, δ^*) is double fuzzy proximity space if it satisfies:

(LP) $\delta(f, g) = \delta(g, f)$ and $\delta^*(f, g) = \delta^*(g, f)$.

Proposition 3.2

(1) *Let (X, η, η^*) be a double fuzzy (resp. symmetric) topogenous space and let the maps $\delta_\eta, \delta^*_{\eta^*} : L^X \times L^X \to L$ defined by $\delta_\eta(f, g) = (\eta(f, g^*))^*$ and* $\delta^*_{\eta^*}(f, g) =$

$(\eta^*(f,g^*))^*$, $\forall f,\ g \in L^X$. *Then* $(\delta_\eta, \delta^*_{\eta^*})$ *is double fuzzy quasi-proximity space (resp. double fuzzy proximity space) on X.*

(2) *Let* (δ, δ^*) *be a double fuzzy quasi-proximity space(resp. double fuzzy proximity space) on and let the mappings* $\eta_\delta, \eta^*_{\delta^*} : L^X \times L^X \to L$ *defined by* $\eta_\delta(f,g) = (\delta(f,g^*))^*$ *and* $\eta^*_{\delta^*}(f,g) = (\delta^*(f,g^*))^*$, $\forall f,\ g \in L^X$. *Then* $(\eta_\delta, \eta^*_{\delta^*})$ *is double fuzzy (resp. symmetric) topogenous space.*

(3) $(\delta, \delta^*) = (\delta_{\eta_\delta}, \delta^*_{\eta^*_{\delta^*}})$*and*$(\eta_{\delta_\eta}, \eta^*_{\delta^*_{\eta^*}}) = (\eta, \eta^*)$.

Proof. (1) Since $\eta \circ \eta \geqslant \eta$ and $\eta^* \circ \eta^* \leqslant \eta^*$.

$$\begin{aligned}
\delta_\eta(f,g) &= (\eta(f,g^*))^* \geqslant ((\eta\circ\eta)(f,g^*))^* \\
&\geqslant (\bigvee_{h\in L^X}\{(\eta(f,h)\odot\eta(h,g^*)\})^* \\
&= \bigwedge_{h\in L^X}\{(\eta(f,h))^* \oplus (\eta(h,g))^*\}\Psi \\
&= \bigwedge_{h\in L^X}\{\delta_\eta(f,h^*) \oplus \delta_\eta(h,g)\},
\end{aligned}$$

$$\begin{aligned}
\delta^*_{\eta^*}(f,g) &= (\eta^*(f,g^*))^* \leqslant ((\eta^*\circ\eta^*)(f,g^*))^* \\
&\leqslant (\bigwedge_{h\in L^X}\{(\eta^*(f,h) \oplus \eta^*(h,g^*)\})^* \\
&= \bigvee_{h\in L^X}\{(\eta^*(f,h))^* \odot (\eta(h,g))^*\} \\
&= \bigvee_{h\in L^X}\{\delta^*_{\eta^*}(f,h^*) \odot \delta^*_{\eta^*}(h,g)\}.
\end{aligned}$$

Let (X,δ,δ^*) be a double fuzzy symmetric topogenous space. Then

$$\delta_\eta(f,g) = (\eta(f,g^*))^* = (\eta(g,f^*)^* = \delta_\eta(f,g),$$
$$\delta^*_{\eta^*}(f,g) = (\eta^*(f,g^*))^* = (\eta^*(g,f^*)^* = \delta^*_{\eta^*}(f,g).^{\Psi}$$

Others are easily proved. □

(2) and (3) are easily proved.

From Theorem 2.9 and 10, we can obtain the following theorems.

Theorem 3.3. *Let* (δ, δ^*) *be a double fuzzy quasi-proximity space. The mapping* $\Psi C_{\delta,\delta^*} : L^X \times L_0 \times L_1 \to L^X$ *defined by*

$$\Psi C_{\delta,\delta^*}(f,r,s) = \wedge\{g^* \in L^X : \delta(g,f)r^* \quad \text{and } \delta(g,f) \geqslant s\}.$$

For each $f, f_1, f_2 \in L^X$, $r, r_1 \in L_0$ and $s, s_1 \in L_1$, we have the following properties:

(1) $C_{\delta,\delta^*}(1_\emptyset, r, s) = 1_\emptyset$.
(2) $f \leqslant C_{\delta,\delta^*}(f,r,s)$.
(3) If $f_1 \leqslant f_2$, *then* $C_{\delta,\delta^*}(f_1,r,s) \leqslant C_{\delta,\delta^*}(f_2,r,s)$.
(4) $C_{\delta,\delta^*}(f_1 \oplus f_2, r\odot r_1, s \oplus s_1) \leqslant C_{\delta,\delta^*}(f_1,r,s) \oplus C_{\delta,\delta^*}(f_2,r_1,s_1)$.
(5) *If* $r \leqslant r_1$ *and* $s \geqslant s_1$ *then* $C_{\delta,\delta^*}(f,r,s) \leqslant C_{\delta,\delta^*}(f,r_1,s_1)$.
(6) $C_{\delta,\delta^*}(C_{\delta,\delta^*}(f,r,s),r,s) \leqslant C_{\delta,\delta^*}(f,r,s)$.

Theorem 3.4. *Let* (X,δ,δ^*) *be a double fuzzy quasi-proximity space. Define the maps* $\mathcal{T}_\delta, \mathcal{T}^*_{\delta^*} : L^X \to L$ *by*

$$\mathcal{T}_\delta(f) = \vee\{r \in L_0 : C_{\delta,\delta^*}(f^*,r,s) \leqslant f^*\},$$
$$\mathcal{T}^*_{\delta^*}(f) = \wedge\{s \in L_1 : C_{\delta,\delta^*}(f^*,r,s) \leqslant f^*\}.$$

Then $(\mathcal{T}_\delta, \mathcal{T}^*_{\delta^*})$ *is double fuzzy topology on induced by* (δ,δ^*).

Definition 3.5. Let (X,δ_1,δ_1^*) and (Y,δ_2,δ_2^*) be double fuzzy quasi-proximity spaces. A map $\phi : (X,\delta_1,\delta_1^*) \to (Y,\delta_2,\delta_2^*)$ is said to be quasi-proximity continuous if

$$\begin{aligned}
&\delta_2(f,g) \geqslant \delta_1(\phi_L^{\leftarrow}(f), \phi_L^{\leftarrow}(g)) \quad \text{or } \delta_2^*(f,g) \\
&\qquad \leqslant \delta_1^*(\phi_L^{\leftarrow}(f), \phi_L^{\leftarrow}(g))\forall f, \quad g \in L^Y.
\end{aligned}$$

Theorem 3.6. *Let* (X,δ_1,δ_1^*) *and*(Y,δ_2,δ_2^*) *be double fuzzy quasi-proximity spaces. A map* $\phi : (X,\delta_1,\delta_1^*) \to (Y,\delta_2,\delta_2^*)$ *is quasi-proximity continuous iff* $\phi : (X,\eta_{\delta_1},\eta^*_{\delta_1^*}) \to (Y,\eta_{\delta_2},\eta^*_{\delta_2^*})$ *is topogenous continuous.*

Proof. For all $f,\ g \in L^Y$

$$\begin{aligned}
\delta_2(f,g) \geqslant \delta_1(\phi_L^{\leftarrow}(f), \phi_L^{\leftarrow}(g)) &\Longleftrightarrow (\eta_{\delta_2}(f,g^*))^* \geqslant (\eta_{\delta_1}(\phi_L^{\leftarrow}(f), (\phi_L^{\leftarrow}(g))^*)^* \\
&\Longleftrightarrow \eta_{\delta_2}(f,g^*) \leqslant \eta_{\delta_1}(\phi_L^{\leftarrow}(f), \phi_L^{\leftarrow}(g^*)), \\
\delta_2^*(f,g) \leqslant \delta_1^*(\phi_L^{\leftarrow}(f), \phi_L^{\leftarrow}(g)) &\Longleftrightarrow (\eta^*_{\delta_2^*}(f,g^*))^* \leqslant (\eta^*_{\delta_1^*}(\phi_L^{\leftarrow}(f), (\phi_L^{\leftarrow}(g))^*)^* \\
&\Longleftrightarrow \eta^*_{\delta_2^*}(f,g^*) \geqslant \eta^*_{\delta_1^*}(\phi_L^{\leftarrow}(f), \phi_L^{\leftarrow}(g^*)). \quad \square
\end{aligned}$$

4. Double fuzzy quasi-uniform spaces and double fuzzy syntopogenous spaces

Now we recall some notions and terminologies about double fuzzy quasi-uniform spaces used in this paper.

Let $\Omega(L^X)$ denote the family of all mappings $a : L^X \to L^X$ with the following properties:

(1) $f \leqslant a(f)$ *for each* $f \in L^X$,
(2) $a(\bigvee_{i\in\Gamma} f_i) = \bigvee_{i\in\Gamma} a(f_i)$, *for each* $f_i \in L^X$.

For $a,\ b \in \Omega(L^X)$, we have that a^{-1}, $a \odot$ b and $a \circ b \in \Omega(L^X)$ by

$$a^{-1}(f) = \wedge\{g \mid a(g^*) \leqslant f^*\},$$
$$(a\odot b)(f) = \wedge\{a(f_1) \oplus b(f_2) \mid f_1 \oplus f_2 = f\} \text{ and } (a\circ b)(f) = a(b(f)).$$

Definition 4.1. The mappings $\mathcal{U}, \mathcal{U}^* : \Omega(L^X) \to L$ is called a double fuzzy quasi-uniformity on X if it satisfies the following conditions: for $a,\ b \in \Omega(L^X)$,

(LU1) $\mathcal{U}(a) \leqslant (\mathcal{U}^*(a))^*$, for all $a \in \Omega(L^X)$,
(LU2) $\mathcal{U}(a\odot b) \geqslant \mathcal{U}(a) \odot \mathcal{U}(b)$ and $\mathcal{U}^*(a\odot b) \leqslant \mathcal{U}^*(a) \oplus \mathcal{U}^*(b)$.
(LU3) *there exists* $a \in \Omega(L^X)$ such that $\mathcal{U}(a) = \top$ and $\mathcal{U}^*(a) = \perp$
(LU4) $\mathcal{U}(a) \leqslant \vee\{\mathcal{U}(b) : b\circ b \leqslant a\}$ and $\mathcal{U}^*(b) \geqslant \{\mathcal{U}^*(b) : b\circ b \leqslant a\}$.

The triple $(X,\mathcal{U},\mathcal{U}^*)$ is said to be double fuzzy quasi-uniform space.

A double fuzzy quasi-uniform space $(X,\mathcal{U},\mathcal{U}^*)$ is said to be a double fuzzy uniform space if it satisfies.

(LU) $\mathcal{U}(a) = \mathcal{U}(a^{-1})$ and $\mathcal{U}^*(a) = \mathcal{U}^*(a^{-1})$.

Definition 4.2. The mappings $\mathcal{B}, \mathcal{B}^* : \Omega(L^X) \to L$ is called a double fuzzy quasi-uniform base on X if it satisfies the following conditions: for $a,\ b \in \Omega(L^X)$,

(LUB1) $\mathcal{B}(a) \leqslant (\mathcal{B}^*(a))^*$, for all $a \in \Omega(L^X)$,
(LUB2) $\mathcal{B}(a) \odot \mathcal{B}(b) \geqslant \vee\{\mathcal{B}(b) : b \leqslant a \odot b\}$ and $\mathcal{B}^*(a) \oplus \mathcal{B}^*(b) \leqslant \wedge\{\mathcal{B}^*(b) : b \leqslant a \odot b\}$.
(LUB3) there exists $a \in \Omega(L^X)$ such that $\mathcal{B}(a) = \top$ and $\mathcal{B}^*(a) = \perp$
(LUB4) $\mathcal{B}(a) \leqslant \vee\{\mathcal{B}(b) : b \circ b \leqslant a\}$ and $\mathcal{B}^*(a) \geqslant \wedge \{\mathcal{B}^*(b) : b \circ b \leqslant a\}$.

A double fuzzy quasi-uniform base $(\mathcal{B}, \mathcal{B}^*)$ on X is said to be double fuzzy uniform base if it satisfies

(LUB) $\mathcal{B}(a) \leqslant \vee\{\mathcal{B}(b) : b \leqslant a^{-1}\}$ and $\mathcal{B}^*(a) \geqslant \wedge\{\mathcal{B}^*(b) : b \leqslant a^{-1}\}$.

Theorem 4.3. *Let $\mathcal{B}, \mathcal{B}^* : \Omega(L^X) \to L$ be a double fuzzy uniform base on X.*

Define $\mathcal{U}_{\mathcal{B}}, \mathcal{U}^*_{\mathcal{B}^*} : \Omega(L^X) \to L$ as

$\mathcal{U}_{\mathcal{B}}(a) = \vee\{\mathcal{B}(b) : b \leqslant a\}$ and $\mathcal{U}^*_{\mathcal{B}^*}(a) = \wedge\{\mathcal{B}^*(b) : b \leqslant a\}$.

Then $(\mathcal{U}_{\mathcal{B}}, \mathcal{U}^*_{\mathcal{B}^*})$ is double fuzzy uniformity on it X.

Proof. (LU) For each $\Psi a \in \Omega(L^X)$

$$\mathcal{U}_{\mathcal{B}}(a) = \vee\{\mathcal{B}(b) : b \leqslant a\} \leqslant \bigvee_{b\leqslant a}\{\bigvee_{c\leqslant b^{-1}} \mathcal{B}(c)\} \qquad (by LUB)$$
$$\leqslant \bigvee_{b\leqslant a} \mathcal{U}_{\mathcal{B}}(b^{-1}) = \bigvee_{b^{-1}\leqslant a^{-1}} \mathcal{U}_{\mathcal{B}}(b^{-1}) \leqslant \mathcal{U}_{\mathcal{B}}(a^{-1}),$$
$$\mathcal{U}^*_{\mathcal{B}^*}(a) = \wedge\{\mathcal{B}^*(b) : b \leqslant a\} \geqslant \bigwedge_{b\leqslant a}\{\bigwedge_{c\leqslant b^{-1}} \mathcal{B}(c)\} \qquad (by LUB)$$
$$\geqslant \bigwedge_{b\leqslant a} \mathcal{U}^*_{\mathcal{B}^*}(b^{-1}) = \bigwedge_{b^{-1}\leqslant a^{-1}} \mathcal{U}^*_{\mathcal{B}^*}(b^{-1}) \geqslant \mathcal{U}^*_{\mathcal{B}^*}(a^{-1}).$$

Since $a = (a^{-1})^{-1}$, we have $\mathcal{U}_{\mathcal{B}}(a^{-1}) \leqslant \mathcal{U}_{\mathcal{B}}(a)$ and $\mathcal{U}^*_{\mathcal{B}^*}(a^{-1}) \geqslant \mathcal{U}^*_{\mathcal{B}^*}(a)$. □

Other cases are easily proved.

Definition 4.4. Let Υ_X be a double fuzzy biperfect syntopogenous structure on X. The mappings $\mathcal{S}, \mathcal{S}^* : \Upsilon_X \to L$ is called double fuzzy syntopogenous structure on X if it satisfies the following conditions: for for (η, η^*), (η_1, η_1^*), $(\eta_2, \eta_2^*) \in \Upsilon_X$.

(LT1) $\mathcal{S}(\eta, \eta^*) \leqslant (\mathcal{S}^*(\eta, \eta^*))^*$.
(LT2) There exists $(\eta, \eta^*) \in \Upsilon$ such that $\mathcal{S}(\eta, \eta^*) = \top$ and $\mathcal{S}^*(\eta, \eta^*) = \perp$
(LT3) $\mathcal{S}(\eta_1, \eta_1^*) \odot \mathcal{S}(\eta_2, \eta_2^*) \leqslant \vee\{\mathcal{S}(\eta, \eta^*) : \ \eta_1, \eta_2 \leqslant \eta$ and $\eta_1^*, \eta_2^* \geqslant \eta^*\}$ and $\mathcal{S}^*(\eta_1, \eta_1^*) \oplus \mathcal{S}^*(\eta_2, \eta_2^*) \geqslant \wedge\{\mathcal{S}^*(\eta, \eta^*) : \eta_1, \eta_2 \leqslant \eta$ and $\eta_1^*, \eta_2^* \geqslant \eta^*\}$.
(LT4) $\mathcal{S}(\eta, \eta^*) \leqslant \vee\{\mathcal{S}(\eta_1, \eta_1^*) : \eta_1 \circ \eta_2 \geqslant \eta$ and $\eta_1^* \circ \eta_2^* \leqslant \eta^*\}$ and $\mathcal{S}^*(\eta_1, \eta_1^*) \geqslant \wedge\{\mathcal{S}^*(\eta_1, \eta_1^*) : \eta_1 \circ \eta_2 \geqslant \eta$ and $\eta_1^*, \eta_2^* \leqslant \eta^*\}$.

The triple $(X, \mathcal{S}, \mathcal{S}^*)$ is said to be double fuzzy syntopogenous space.

A double fuzzy syntopogenous space $(X, \mathcal{S}, \mathcal{S}^*)$ is said to be double fuzzy symmetric syntopogenous space if it satisfies

(LST) , $\mathcal{S}(\eta, \eta^*) \leqslant \vee\{\mathcal{S}(\xi, \xi^*) : \xi \geqslant \eta^s$ and $\xi^* \leqslant \eta^{*s}\}$,

$\mathcal{S}^*(\eta, \eta^*) \geqslant \wedge\{\mathcal{S}^*(\xi, \xi^*) : \xi \geqslant \eta^s$ and $\xi^* \leqslant \eta^{*s}\}$.

Lemma 4.5. *To every $a \in \Omega(L^X)$, we define $\eta_a, \ \eta_a^* : L^X \times L^X \to L$ as*

$$\eta_a(g,f) = \begin{cases} \top, & \text{if } f \geqslant a(g), \\ \perp & \text{otherwise}, \end{cases} \quad \eta_a^*(g,f) = \begin{cases} \perp, & \text{if } f \geqslant a(g), \\ \top & \text{otherwise}. \end{cases}$$

Then it satisfies the following properties:

(1) *The maps $\eta_a, \ \eta_a^* \in \Upsilon_X$ is double fuzzy biperfect topogenous order.*
(2) *If $a \leqslant b$, then $\eta_b \leqslant \eta_a$ and $\eta_b^* \leqslant \eta_a^*$.*
(3) *If $b \leqslant a_1 \odot a_2$, then $\eta_{a_1}, \ \eta_{a_2} \leqslant \eta_b$ and $\eta_{a_1}^*, \ \eta_{a_2}^* \geqslant \eta_b^*$.*
(4) *For each $a \in \Omega(L^X)$, we have $(\eta_a^s, \eta_a^{*s}) = (\eta_{a^{-1}}, \eta_{a^{-1}}^*)$.*
(5) *If $b \circ b \leqslant a$, then $\eta_b \circ \eta_b \geqslant \eta_a$* and *$\eta_b^* \circ \eta_b^* \leqslant \eta_a^*$.*

Proof. (1) Since $\Psi a(1_X) = 1_X$ and $a(1_\emptyset) = 1_\emptyset$, then $\eta_a(1_X, 1_X) = \eta_a(1_\emptyset, 1_\emptyset) = \top\Psi$ and $\eta_a^*(1_X, 1_X) = \eta_a^*(1_\emptyset, 1_\emptyset) = \perp$ Let $\eta_a(g,f) \neq \perp$ and $\eta_a^*(g,f) \neq \top$. Then $\eta_a(g,f) = \top$ and $\eta_a^*(g,f) = \perp$ implies $g \leqslant a(g) \leqslant f$. Since $g \leqslant g_1$ and $f_1 \leqslant f$ implies $a(g) \leqslant a(g_1)$, then$\eta_a(g_1, f_1) \leqslant \eta_a(g,f)$ and $\eta_a^*(g_1, f_1) \geqslant \eta_a^*(g,f)$. To prove the biperfect condition, since $a(\bigvee_{i\in\Gamma} g_i) = \vee_{i\in\Gamma} a(g_i) \leqslant f$ iff $g_i \leqslant f$ for all $i \in \Gamma, \eta_a(\bigvee_{i\in\Gamma} g_i, f) \geqslant \bigwedge_{i\in\Gamma} \eta_a(g_i, f)$ and $\eta_a^*(\bigvee_{i\in\Gamma} g_i, f) \leqslant \bigvee_{i\in\Gamma} \eta_a^*(g_i, f)$. Since $g \leqslant \bigwedge_{i\in\Gamma} f_j$ iff $g \leqslant f_j$ for all $j \in \Lambda$, $\eta_a(g, \bigwedge_{j\in\Lambda} f_j) \geqslant \bigwedge_{j\in\Lambda} \eta_a(g, f_j)$ and $\eta_a^*(g, \bigwedge_{j\in\Lambda} f_j) \leqslant \bigvee_{j\in\Lambda} \eta_a^*(g, f_j)$.

Others are similarly proved.

(2) Since $\Psi a(g) \leqslant b(g)$, $\eta_b \leqslant \eta_a$ and $\eta_b^* \geqslant \eta_a^*$.

(3) Since $a_1 \odot a_2(g) \leqslant a_1(g) \oplus a_2(g)\Psi$ we have $a_1 \odot a_2 \leqslant a_1$. From (2), $\eta_{a_1} \leqslant \eta_b$ and $\eta_{a_1}^* \geqslant \eta_b^*$. Similary $\eta_{a_2} \leqslant \eta_b$ and $\eta_{a_2}^* \geqslant \eta_b^*.\Psi$

(4) It easily proved from $a^{-1}(g) \leqslant f$ iff $a(f^*) \leqslant g^*$.

(5) From (2), we only show that $\eta_b \odot \eta_b = \eta_{b\circ b}$ and $\eta_b^* \odot \eta_b^* = \eta_{b\circ b}^*$. Since $\eta_b \circ \eta_b(g,f) = \vee\{\eta_b(g,h) \odot \eta_b(h,f) : h \in L^X\}and\eta_b^* \ \circ \eta_b^*(g,f) = \wedge\{\eta_b^*(g,h) \ \oplus \ \eta_b^*(h,f) : h \in L^X\}$, we have

$$\eta_b \circ \eta_b(g,f) = \begin{cases} \top \text{ if } f \geqslant b(b(g)), \\ \perp \text{ otherwise}, \end{cases} \eta_b^* \circ \eta_b^*(g,f) = \begin{cases} \perp \text{ if } f \geqslant b(b(g)), \\ \top \text{otherwise}. \end{cases}$$

From Lemma 4.5, we easily prove the following theorem. □

Theorem 4.6. *Let $\mathcal{B}, \mathcal{B}^* : \Omega(L^X) \to L$ be double fuzzy quasi-uniform (resp. double fuzzy uniform) base on X. Define $\mathcal{S}_{\mathcal{B}}, \mathcal{S}^*_{\mathcal{B}^*} : \Upsilon_X \to L$ as*

$\mathcal{S}_{\mathcal{B}}(\eta_a, \eta_{a^*}^*) = \mathcal{B}(a)$ and $\mathcal{S}^*_{\mathcal{B}^*}(\eta_a, \eta_{a^*}^*) = \mathcal{B}^*(a)$.

Then $(\mathcal{S}_{\mathcal{B}}, \mathcal{S}^*_{\mathcal{B}^*})$ is double fuzzy (resp. double fuzzy symmetric) syntopogenous structure on X.

Theorem 4.7. *Let $\mathcal{U}, \mathcal{U}^* : \Omega(L^X) \to L$ be a double fuzzy quasi-uniformity on X.*

The mapping $\mathcal{C}_{\mathcal{U}, \mathcal{U}^*} : L^X \times L_0 \times L_1 \to L^X$, is defined by

$\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s) = \wedge\{a(f) : \mathcal{U}(a) \geqslant r \text{ and } \mathcal{U}^*(a) \leqslant s\}$.

For each f, f_1, $f_2 \in L^X$, r, r_1, $r_2 \in L_0$ and s, s_1, $s_2 \in L_1$, we have the following properties:

(1) $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(1_\emptyset,r,s) = 1_\emptyset$,
(2) $f \leqslant \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s)$,
(3) if$f_1 \leqslant f_2$, then $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_1,r,s) \leqslant \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_2,r,s)$,
(4) $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_1 \oplus f_2, r\odot r_1, s \oplus s_1) \leqslant \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_1,r,s) \oplus \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_2,r_1,s_1)$,
(5) if$r_1 \leqslant r_2$ and $s_1 \geqslant s_2$, then $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_1,r_1,s_1) \leqslant \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_2,r_2,s_2)$,
(6) $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s),r,s) \leqslant \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s)$.

Proof

(1) Since $\Psi a(1_\emptyset) = 1_\emptyset, \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(1_\emptyset,r,s) = 1_\emptyset$.
(2) Since $\Psi f \leqslant a(f)$, Ψ implies $\Psi f \leqslant \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s)$,
(3) and (5) are easily proved.
(4) Conversely, suppose there exist f, f_1, $f_2 \in L^X$, r, r_1, $r_2 \in L_0$ and s, s_1, $s_2 \in L_1$, such that

$\leftarrow \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_1 \oplus f_2, r\odot r_1, s \oplus s_1) \not\leqslant \rightarrow \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_1,r,s) \oplus \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_2,r_1,s_1)$

There exist $a_1,a_2 \in \Omega(L^X)$ with $\mathcal{U}(a_1) \geqslant r$, $\mathcal{U}^*(a_1) \leqslant s$, $\mathcal{U}(a_2) \geqslant r_1$, $\mathcal{U}^*(a_2) \leqslant s_1$, such that $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_1 \oplus f_2, r\odot r_1, s \oplus s_1)\,\overline{a}(f_1) \oplus a(f_2)$.

On the other hand, $\mathcal{U}(a_1 \odot a_2) \geqslant \mathcal{U}(a_1)\odot\ \mathcal{U}(a_2) \geqslant r\odot r_1, \mathcal{U}^*\ (a_1 \odot a_2) \leqslant \mathcal{U}^*(a_1) \oplus \mathcal{U}^*(a_2) \leqslant s \oplus s_1$ and $(a_1 \odot a_2)(f_1 \oplus f_2) \leqslant a_1(f_1) \oplus a_2(f_2)$, we have $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_1 \oplus f_2, r\odot r_1, s \oplus s_1) \leqslant (a_1 \odot a_2)(f_1 \oplus f_2) \leqslant a_1(f_1) \oplus a_2(f_2)$. It is a contradiction.

(6) Suppose there exist $\Psi f \in L^X$, $r \in L_0$ and $s \in L_1$ such that $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s),r,s) \not\leqslant\ \leftarrow \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s)$.

There exists $a \in \Omega(L^X)\Psi$ with $\mathcal{U}(a) \geqslant r$ and $\mathcal{U}^*(a) \leqslant s$, $\leftarrow$ such that $\leftarrow\ \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s) \leqslant a(f)$. On the other hand, $\mathcal{U}(a) \geqslant r$ and $\mathcal{U}^*(a) \leqslant s$, by (LSU3), there exists $a_1 \in \Omega(L^X)$ $\leftarrow$ such that $a_1 \circ a_2 \leqslant a, \mathcal{U}(a_1) \geqslant r$ and $\mathcal{U}^*(a_1) \leqslant s$. Ψ Since $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s) \leqslant a_1(f)$, Ψ we have

$\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s),r,s) \leqslant \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(a_1(f),r,s) \leqslant a_1(a_1(f)) \leqslant a(f)$.

Thus $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s),r,s) \leqslant a(f)$. It is a contradiction. □

Definition 4.8. Let $(X,\mathcal{U},\mathcal{U}^*)$ and $(Y,\mathcal{V},\mathcal{V}^*)$ be double fuzzy uniform (resp. double fuzzy quasi-uniform) spaces. A mapping $\Psi\Psi\ \phi : (X,\mathcal{U},\mathcal{U}^*) \to (Y,\mathcal{V},\mathcal{V}^*)$ is said to be uniformly continuous (resp. quasi-uniformly continuous) if

$$\mathcal{V}(a) \leqslant \mathcal{U}(\phi_L^{\leftarrow}(a)) \text{ and } \mathcal{V}^*(a) \geqslant \mathcal{U}^*(\phi_L^{\leftarrow}(a)), \quad \forall\ a \in \Omega_X,$$

where $\Phi\ \phi_L^{\leftarrow}(a)(f) = \phi_L^{\leftarrow}(a(\phi_L^{\rightarrow}(f)))$ for all $f \in L^X$.

From Theorem 4.3, we easily prove the following theorem.

Definition 4.9. Let $(X,\mathcal{B}_1,\mathcal{B}_1^*)$ and $(Y,\mathcal{B}_1,\mathcal{B}_1^*)$ be double fuzzy quasi-uniform bases. If $\mathcal{B}_2(a) \leqslant \mathcal{B}_1(\phi_L^{\leftarrow}(a))$ and $\mathcal{B}_2^*(a) \geqslant \mathcal{B}_1^*(\phi_L^{\leftarrow}(a))$ for all $a \in \Omega(L^Y)$ then $\phi :\ (X,\mathcal{U}_{\mathcal{B}_1}, \mathcal{U}^*_{\mathcal{B}_1^*}) \to (Y,\mathcal{U}_{\mathcal{B}_2},\mathcal{U}^*_{\mathcal{B}_2^*})$ is quasi-uniformly continuous.

Theorem 4.10. *Let* $(X,\mathcal{U},\mathcal{U}^*),(Y,\mathcal{V},\mathcal{V}^*)$ and $(Z,\mathcal{W},\mathcal{W}^*)$ *be double fuzzy quasi-uniform spaces. If* $\phi : (X,\mathcal{U},\mathcal{U}^*) \to (Y,\mathcal{V},\mathcal{V}^*)$ and $\psi : (Y,\mathcal{V},\mathcal{V}^*) \to (Z,\mathcal{W},\mathcal{W}^*)$ *are quasi-uniformly continuous, then* $\psi\circ\phi : (X,\mathcal{U},\mathcal{U}^*) \to (Z,\mathcal{W},\mathcal{W}^*)$ *is quasi-uniformly continuous.*

Theorem 4.11. *Let* $(X,\mathcal{U},\mathcal{U}^*)and(Y,\mathcal{V},\mathcal{V}^*)$ *be double fuzzy quasi-uniform spaces. Let* $\phi : (X,\mathcal{U},\mathcal{U}^*) \to (Y,\mathcal{V},\mathcal{V}^*)$ *be quasi-uniformly continuous. Then:*

(1) $\phi_L^{\rightarrow}(\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s)) \leqslant \mathcal{C}_{\mathcal{V},\mathcal{V}^*}(\phi_L^{\rightarrow}(f),r,s)$, *for each* $f \in L^X$, $r \in L_0$, $s \in L_1$.
(2) $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(\phi_L^{\leftarrow}(g),r,s) \leqslant \phi_L^{\leftarrow}(\mathcal{C}_{\mathcal{V},\mathcal{V}^*}(g,r,s))$, *for each* $g \in L^y$, $r \in L_0$, $s \in L_1$.
(3) $\phi : (X,\mathcal{T}_{\mathcal{U}},\mathcal{T}^*_{\mathcal{U}^*}) \to (Y,\mathcal{T}_{\mathcal{V}},\mathcal{T}^*_{\mathcal{V}^*})$ *is fuzzy continuous.*

Proof. (1) Suppose there exist $f \in L^X$, $r \in L_0$ and $s \in L_1$ such that

$\phi_L^{\rightarrow}(\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s)) \not\leqslant \mathcal{C}_{\mathcal{V},\mathcal{V}^*}(\phi_L^{\rightarrow}(f),r,s)$.

There exists $a \in \Omega(L^Y)$ with $\mathcal{V}(a) \geqslant r$ and $\mathcal{V}^*(a) \leqslant s$ such that $\mathcal{C}_{\mathcal{V},\mathcal{V}^*}(\phi_L^{\rightarrow}(f),r,s) \not\leqslant \overline{a}(\phi_L^{\rightarrow}(f))$.

On the other hand, ϕ is quasi-uniformly continuous,

$$\mathcal{U}(\phi_L^{\leftarrow}(a)) \geqslant \mathcal{V}(a) \geqslant r \quad \text{and } \mathcal{U}^*(\phi_L^{\leftarrow}(a)) \leqslant \mathcal{V}^*(a) \leqslant s.$$

It implies $a(\phi_L^{\rightarrow}(f))(\phi(x)) = \phi_L^{\leftarrow}(a)(f)(x) \geqslant \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r)(x)$. It is a contradiction.

(2) and (3) are similarly proved as Theorem 2.13. □

Theorem 4.12. *Let* $(X,\mathcal{S},\mathcal{S}^*)$ *be a double fuzzy syntopogenous space. The mapping* $\mathcal{C}_{\mathcal{S},\mathcal{S}^*} : L^X \times L_0 \times L_1 \to L^X$, *is defined by*

$$\mathscr{C}_{\mathscr{S},\mathscr{S}^*}(f,r,s) = \wedge\{g : \eta(f,g)\ >\perp, \eta^*(f,g) \leqslant \top, \mathscr{S}(\eta,\eta^*) \geqslant r \text{ and } \mathscr{S}^*(\eta,\eta^*) \leqslant s\}.$$

For each Ψf, f_1, $f_2 \in L^X$, r, r_1, $r_2 \in L_0$ and s, s_1, $s_2 \in L_1$,we have the following properties:

(1) $\mathcal{C}_{\mathcal{S}},\mathcal{S}^*(1_\emptyset,r,s) = 1_\emptyset$,
(2) $f \leqslant \mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f,r,s)$,
(3) if $f_1 \leqslant f_2, \mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f_1,r,s) \leqslant \mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f_2,r,s)$,
(4) $\mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f_1 \oplus f_2, r\odot r_1, s \oplus s_1) \leqslant \mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f_1,r,s) \oplus \mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f_2,r_1,s_1)$,
(5) if $r_1 \leqslant r_2$ *and* $s_1 \geqslant s_2$, *then* $\mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f,r_1,s_1) \leqslant \mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f,r_2,s_2)$,
(6) $\mathcal{C}_{\mathcal{S}},\mathcal{S}^*(\mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f,r,s),r,s) = \mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f,r,s)$.

Proof. (1) Since Ψ $\eta(1_\emptyset,1_\emptyset) = \top$ and $\eta^*(1_\emptyset,1_\emptyset) =\perp$ for $\mathcal{S}(\eta,\eta^*) = \top, \mathcal{S}^*(\eta,\eta^*) = \perp$,

$$\mathcal{C}_{\mathcal{S}},\mathcal{S}^*(1_\emptyset,r,s) = 1_\emptyset,$$

(2) Since $\Psi f \leqslant g$ for $\eta(f,g) \perp$ and $\eta^*(f,g) \leqslant \top$, $f \leqslant \mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f,r,s)$.

(3) and (5) are easily proved.

(4) Suppose there exist $f_1, f_2, \in L^X$, r, $r_1 \in L_0$ and s, $s_1 \in L_1$ such that

$\mathcal{C}_{\mathcal{S}}, \mathcal{S}^*(f_1 \oplus f_2, r \odot r_1, s \oplus s_1) \not\leqslant \mathcal{C}_{\mathcal{S}}, \mathcal{S}^*(f_1, r, s)$
$\oplus \mathcal{C}_{\mathcal{S}}, \mathcal{S}^*(f_2, r_1, s_1)$.

There exist,

$(\eta_1, \eta_1^*), (\eta_2, \eta_2^*) \in \Upsilon_X$ with $\mathcal{S}(\eta_1, \eta_1^*) \geqslant r, \mathcal{S}(\eta_2, \eta_2^*) \geqslant r_1, \mathcal{S}^*(\eta_1, \eta_1^*) \leqslant s$, $\mathcal{S}^*(\eta_2, \eta_2^*) \leqslant s_1, \eta_i(f_i, g_i) \not\leqslant \perp$ and $\eta_1^*(f_i, g_i) \leqslant \top$ such that, $\mathcal{C}_{\mathcal{S},\mathcal{S}^*}(f_1 \oplus f_2, r \odot r_1, s \oplus s_1) \not\leqslant g_1 \oplus g_2$.

On the other hand, $\mathcal{S}(\eta_1, \eta_1^*) \odot \mathcal{S}(\eta_2, \eta_2^*) \geqslant r$ and $\mathcal{S}^*(\eta_1, \eta_1^*) \odot \mathcal{S}^*(\eta_2, \eta_2^*) \leqslant s$, by (LT2) of Definition 4.4, there exist $\Psi\eta \geqslant \eta_i, \eta^* \leqslant \eta_i^*, \mathcal{S}(\eta, \eta^*) \geqslant r$ and $\mathcal{S}^*(\eta, \eta^*) \leqslant s$ such that

$$\begin{aligned}\eta(f_1 \oplus f_2, g_1 \oplus g_2) &\geqslant \eta(f_1, g_1 \oplus g_2) \odot \eta(f_2, g_1 \oplus g_2)\\ &\geqslant \eta(f_1, g_1) \odot \eta(f_2, g_2)\\ &\geqslant \eta_1(f_1, g_1) \odot \eta_2(f_2, g_2) \not\leqslant \perp.\end{aligned}$$

Hence $C_{\mathcal{S},\mathcal{S}^*}(f_1 \oplus f_2, r \odot r_1, s \oplus s_1) \leqslant g_1 \oplus g_2$. It is a contradiction.

$$\begin{aligned}\eta^*(f_1 \oplus f_2, g_1 \oplus g_2) &\leqslant \eta^*(f_1, g_1 \oplus g_2) \odot \eta^*(f_2, g_1 \oplus g_2)\\ &\leqslant \eta^*(f_1, g_1) \odot \eta^*(f_2, g_2)\\ &\leqslant \eta_1^*(f_1, g_1) \oplus \eta_2^*(f_2, g_2) \leqslant \top.\end{aligned}$$

(6) Suppose there exist $\Psi f \in L^X$, $r \in L_0$ and $s \in L_1$ such that

$$\mathcal{C}_{\mathcal{S},\mathcal{S}^*}(\mathcal{C}_{\mathcal{S},\mathcal{S}^*}(f, r, s), r, s) \not\leqslant \mathcal{C}_{\mathcal{S},\mathcal{S}^*}(f, r, s).$$

There exists $\Psi g \in L^X$ with $\mathcal{S}(\eta, \eta^*) \geqslant r$, $\mathcal{S}^*(\eta, \eta^*) \leqslant s$, $\eta(f, g) \not\leqslant \perp$ and $\eta^*(f, g) \leqslant \top$,
such that $\leftarrow \mathcal{C}_{\mathcal{S},\mathcal{S}^*}(f, r, s) \leqslant g$. On the other hand, $\mathcal{S}(\eta, \eta^*) \not\leqslant r$ and $\mathcal{S}^*(\eta, \eta^*) \leqslant s$, $\leftarrow$ by (LT3) of Definition 4.4, there exists $(\zeta, \zeta^*) \in {}_X$ such that

$$(\zeta, \zeta^*) \circ (\zeta, \zeta^*) \geqslant (\eta, \eta^*), \quad \mathcal{S}(\zeta, \zeta^*) \geqslant r \quad \text{and } \mathcal{S}^*(\zeta, \zeta^*) \leqslant s.$$

Since $\zeta \circ \zeta(f, g) \not\leqslant \perp$ and $\zeta^* \circ \zeta^*(f, g) \leqslant \top$, here exists $\rho \in L^X$ such that $\zeta(f, \rho) \odot \zeta(\rho, g) \not\leqslant \perp$ and $\zeta^*(f, \rho) \oplus \zeta^*(\rho, g) \leqslant \top$. It implies $\mathcal{C}_{\mathcal{S},\mathcal{S}^*}(f, r, s) \leqslant \rho, \mathcal{C}_{\mathcal{S},\mathcal{S}^*}(\rho, r, s) \leqslant g$. Hence $\mathcal{C}_{\mathcal{S},\mathcal{S}^*}(\mathcal{C}_{\mathcal{S},\mathcal{S}^*}(f, r, s), r, s) \leqslant g$. Thus, $\mathcal{C}_{\mathcal{S},\mathcal{S}^*}(\mathcal{C}_{\mathcal{S},\mathcal{S}^*}(f, r, s), r, s) \leqslant g$. It is a contradiction. □

Definition 4.13. Let $(X, \mathcal{S}, \mathcal{S}^*)$ be a double fuzzy syntopogenous space. Define the maps $\mathcal{T}_{\mathcal{S}}, \mathcal{T}^*_{\mathcal{S}} : L^X \to L$ *by*

$$\mathcal{T}_{\mathcal{S}}(f) = \vee\{r \in L_0 : \mathcal{C}_{\mathcal{S},\mathcal{S}^*}(f^*, r, s) \leqslant f^*\},$$
$$\mathcal{T}^*_{\mathcal{S}^*}(f) = \wedge\{s \in L_1 : \mathcal{C}_{\mathcal{S},\mathcal{S}^*}(f^*, r, s) \leqslant f^*\}.$$

Then $(\mathcal{T}_{\mathcal{S}}, \mathcal{T}^*_{\mathcal{S}^*})$ is a double fuzzy topology on Xinduced by $(\mathcal{S}, \mathcal{S}^*)$.

Theorem 4.14. *Let $(X, \mathcal{S}_1, \mathcal{S}_1^*)$ and $(Y, \mathcal{S}_2^*, \mathcal{S}_2^*)$ be double fuzzy syntopogenous space. A map $\phi : (X, \mathcal{S}_1, \mathcal{S}_1^*) \to (Y, \mathcal{S}_2^*, \mathcal{S}_2^*)$ is said to be syntopogenous continuous if for each $(\zeta, \zeta^*) \in {}_Y$, here exists $(\zeta, \zeta^*) \in {}_X$with $\eta(\phi_L^{\leftarrow}(g), \phi_L^{\leftarrow}(f)) \geqslant \zeta(g, f)$ and $\eta^*(\phi_L^{\leftarrow}(g), \phi_L^{\leftarrow}(f)) \leqslant \zeta^*(g, f)$such that $\mathcal{S}_2(\zeta, \zeta^*) \leqslant \mathcal{S}_1(\zeta, \zeta^*)$ and $\mathcal{S}_2^*(\zeta, \zeta^*) \geqslant \mathcal{S}_1^*(\zeta, \zeta^*)$.*

Theorem 4.15. *Let $(X, \mathcal{S}_1, \mathcal{S}_1^*)and(Y, \mathcal{S}_2^*, \mathcal{S}_2^*)$ be double fuzzy syntopogenous space. Let $\phi : (X, \mathcal{S}_1, \mathcal{S}_1^*) \to (Y, \mathcal{S}_2^*, \mathcal{S}_2^*)$ be syntopogenous continuous. Then we have the following properties:*

(1) $\phi_L^{\rightarrow}(\mathcal{C}_{\mathcal{S}_1,\mathcal{S}_1^*}(f, r, s) \leqslant \mathcal{C}_{\mathcal{S}_2,\mathcal{S}_2^*}(\phi_L^{\rightarrow}(f), r, s)$,
$f \in L^X$, $r \in L_0$, *and* $s \in L_1$.
(2) $\mathcal{C}_{\mathcal{S}_1,\mathcal{S}_1^*}(\phi_L^{\leftarrow}(g), r, s) \leqslant \phi_L^{\leftarrow}(\mathcal{C}_{\mathcal{S}_2,\mathcal{S}_2^*}(g, r, s))$,
$g \in L^Y$, $r \in L_0$, *and* $s \in L_1$.
(3) $\phi : (X, \mathcal{T}_{\mathcal{S}_1}, \mathcal{T}^*_{\mathcal{S}_1^*}) \to (Y, ,\mathcal{T}_{\mathcal{S}_2}, \mathcal{T}^*_{\mathcal{S}_2^*})$ *is fuzzy continuous.*

Proof. (1) Suppose there exsit $f \in L^X$, $r \in L_0$, and $s \in L_1$ such that

$$\Psi\phi_L^{\rightarrow}(\mathcal{C}_{\mathcal{S}_1,\mathcal{S}_1^*}(f, r, s) \not\leqslant \mathcal{C}_{\mathcal{S}_2,\mathcal{S}_2^*}(\phi_L^{\rightarrow}(f), r, s).$$

There exists $(\zeta, \zeta^*) \in \Upsilon_X$ with $\mathcal{S}_2(\zeta, \zeta^*) \geqslant r, \mathcal{S}_2^*(\zeta, \zeta^*) \leqslant s, \zeta(\phi_L^{\rightarrow}(x), g) \perp$ and $\zeta^*(\phi_L^{\rightarrow}(x), g) \leqslant \top$ such that $\mathcal{C}_{\mathcal{S}_2,\mathcal{S}_2^*}(\phi_L^{\rightarrow}(f), r, s) \leqslant g$. On the other hand, ϕ is syntopogenous continuous, for each $(\zeta, \zeta^*) \in {}_Y$, there exsits $(\zeta, \zeta^*) \in {}_X$, with $\eta(\phi_L^{\leftarrow}(\phi_L^{\rightarrow}(f)), \phi_L^{\leftarrow}(g)) \geqslant \zeta(\phi_L^{\rightarrow}(f), g)$ and $\eta^*(\phi_L^{\leftarrow}(\phi_L^{\rightarrow}(f)), \phi_L^{\leftarrow}(g)) \leqslant \zeta^*(\phi_L^{\rightarrow}(f), g)$ such that $\mathcal{S}_1(\eta, \eta^*) \geqslant \mathcal{S}_2(\zeta, \zeta^*) \geqslant r$ and $\mathcal{S}_1^*(\eta, \eta^*) \leqslant \mathcal{S}_2^*(\zeta, \zeta^*) \leqslant s$.

It implies $\mathcal{C}_{\mathcal{S}_1,\mathcal{S}_1^*}(f, r, s) \leqslant \phi_L^{\leftarrow}(g)$. It is a contradiction. □

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