

Self-Similar Solutions of Bianchi Type iii Space-Times Using Partial Differential Equations

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ABSTRACT

A different approach is developed to study self-similar solutions in Bianchi type III space-times. Self-similar solutions in both tilted and non tilted cases are obtained. By solving a system of six non linear and inhomogeneous partial differential equations. It is shown that in non-tilted case the above space-times admit orthogonal zeroth kind self-similarity and in tilted case. It admits second and infinite kind self-similarities

1 INTRODUCTION

General theory of relativity is the theory of gravitation which is described in terms of geometry and is highly non-linear. Due to the nonlinearity it is difficult to solve the gravitational field equations unless some symmetry restrictions are imposed on the space-time metric. Over the past few years there has been much interest in studying symmetries in general relativity [Hall, 1998; 1993; Hall et 1990]. These symmetries arise in the exact solutions of Einstein field equations (EFEs) given by

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab},$$

where G_{ab} represents the components of Einstein tensor, R_{ab} are the components of Ricci tensor, T_{ab} are the components of energy momentum tensor and R is the Ricci scalar.

A set of field equations remains invariant under a scale transformation if we assume appropriate matter field. This implies the existence of scale invariant solutions to the Einstein field equations. These solutions are called self-similar solutions. In general relativity self-similarity is defined by the existence of a homothetic vector field. Such similarity is called of the first kind. Self-similar solutions of the Einstein field equations are widely studied for two very important reasons; first, the governing differential equations have some mathematical complexity which is often reduced by the assumption of self-similarity and the system of partial differential equations is reduced to ordinary differential equations. Second, self-similarity solutions are extensively used for cosmological perturbations, star formation, gravitational collapse, primordial black holes, cosmological voids and cosmic censorship [Harada 2004]. Asymptotic behaviour of more general solutions has been discussed [Carr et al 1999; Coley 1999].

Cahill and Taub [Cahill, Taub, 1971] were the first to study perfect fluid similarity solutions in general relativity. They did so in the cosmological context under the assumption of spherical symmetry. They assumed that all dependent variables are functions of a single dimensionless combination of r and t (i.e., the solution is invariant under the transformation $t = \alpha t$ and $r = \alpha r$ for any constant α) and that the model contains no other dimensional constants. This corresponds to the existence of a similarity of the first kind and they showed that it can be invariantly formulated in terms of the existence of a homothetic vector field. Carter and Henriksen [Carter, Henriksen 1989] argue about the self-similarity of the more general second kind i.e. Zeroth Kind. In their second paper [Carter, Henriksen 1989] they gave a basic condition characterizing a vector field as a self-similarity generator, that there exist constants α such that for each physical field Φ_i

$$L_X \Phi^i_A = \alpha \Phi^i_A, \quad (1)$$

where the fields Φ_i can be a scalar, vector e.g. t_a or a tensor e.g. g_{ab} . A.A. Coley [Coley, 1997] discussed the differences between self-similarity of the first kind and generalized self-similarity. He argued that self-similarity is an appropriate generalization of homothety and is the natural relativistic counterpart of self-similarity of the more general second (and zero) kind. He also discussed various mathematical and physical properties of space-times admitting self-similarity. In his work Coley introduced the governing equations for perfect fluid cosmological models and a set of integrability conditions for the existence of proper self-similar vector fields in these models are derived.

Tsampanlis et al (2000) discussed some important symmetries of Bianchi Type I space-times. They determined the self-similarities of Bianchi I metrics without any restriction on the type of the fluid. They showed that Bianchi type I space-times admit self-similarity of first and zeroth kind. Hideki Maeda et al [Maeda et al 2002] classify all spherically symmetric space-times admitting self-similar vector fields of the second, zeroth or infinite kind. They studied the cases in which the self-similar vector field is not only tilted but also parallel or orthogonal to the fluid flow. Sharif and Aziz (2005a, 2005b) worked for self-similarity solutions of spherically and cylindrically symmetric space-times, respectively. In [Sharif, Aziz, 2005] they explored some properties of the self-similar solutions of the first kind for spherically symmetric space-times and also checked the singularities of these solutions. In [Sharif, Aziz, 2005b] they studied the cylindrically symmetric solutions which admit self-similar vector fields of second, zeroth and infinite kinds, for the tilted, parallel and orthogonal cases. They showed that the parallel case gives contradiction both in perfect fluid and dust cases and the orthogonal perfect fluid case yields a vacuum solution while the orthogonal dust case gives contradiction. In all the above papers the authors have first considered the kind of the self-similar solution and then reached to a result whether that specific solution exists or not.

In our case we did not assume any thing. We basically developed algebraic and direct integration technique to get self-similar solutions. In direct integration technique we need to solve the coupled non linear differential equations.

Throughout M represents a four dimensional, connected, hausdorff space-time manifold with Lorentz metric of signature $(-, +, +, +)$. The usual covariant, partial and Lie derivatives are

denoted by a semicolon, a comma and the symbol L respectively. A vector field is said to be Self-Similar if it satisfies the following two conditions [Coley, 1997]:

$$L_X u_a = \alpha u_a, \tag{2}$$

$$L_X h_{ab} = 2\delta h_{ab}, \tag{3}$$

where u^a is the four-velocity of the fluid satisfying $u^a u_a = 1$ and $h_{ab} = g_{ab} - u_a u_b$ is the projection tensor, and $\alpha, \delta \in R$. If $\delta \neq 0$ the similarity transformation is characterized by the scale-independent ratio α/δ , which is called the similarity index. If the ratio is unity, X turns out to be a homothetic vector field. In the context of self-similarity, homothety is referred to as self-similarity of the First kind. If $\alpha = 0$ and $\delta \neq 0$ it is referred to as self-similarity of the Zeroth kind. If the ratio is not equal to zero or one, it is referred to as self-similarity of the Second kind. If $\alpha \neq 0$ and $\delta = 0$ it is referred to as self-similarity of the Infinite kind. If $\alpha = \delta = 0$, X turns out to be a Killing vector fields. It is given that a self-similar vector field can be tilted or non-tilted to the fluid flow. If it is tilted to the fluid flow, then it can be written as $X = (\alpha t + \beta) \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}$. If the self-similar vector field is non tilted then it will be either orthogonal or parallel to the fluid flow. The self-similar vector field in orthogonal case will be $X = f(t) \frac{\partial}{\partial r}$ and in parallel case the self-similar vector field will be $X = F(t) \frac{\partial}{\partial t}$.

2 MAIN RESULTS:

Consider the Bianchi type III space-times in the spherical coordinate system (t, r, θ, ϕ) (labeled by (x^0, x^1, x^2, x^3) respectively) with the line element [Stephani et al 2003]

$$ds^2 = -dt^2 + A(t)dr^2 + B(t)(d\theta^2 + \sinh^2 \theta d\phi^2). \tag{4}$$

The above space-times (4) admit at least four independent killing vector fields [Shabbir and Mehmood, 2007] which are

$$\frac{\partial}{\partial r}, \quad \frac{\partial}{\partial \phi}, \quad \cos\theta \frac{\partial}{\partial \theta} - \cot\theta \sin\phi \frac{\partial}{\partial \phi}, \quad \sin\theta \frac{\partial}{\partial \theta} + \cot\theta \cos\phi \frac{\partial}{\partial \phi}.$$

In this paper we shall take the four-velocity vector as space like and define $u_a = \sqrt{A} \delta_a^1$ where $u^a u_a = 1$ thus the line element (4) becomes

$$ds^2 = -dt^2 + dr^2 + B(t)(d\theta^2 + \sinh^2 \theta d\phi^2). \tag{5}$$

Solving equation (2), we get

$$X^1 = \alpha r + \beta. \tag{6}$$

One can write (3) explicitly using (4)

$$X^0_{,0} = \delta, \tag{7}$$

$$-X^0_{,2} + B(t)X^2_{,0} = 0, \tag{8}$$

$$-X^0_{,3} + B(t)\sinh^2 \theta X^3_{,0} = 0, \tag{9}$$

$$\dot{B} X^0 + 2 B (t) X^2_{,2} = 2 \delta B, \quad (10)$$

$$\sinh^2 \theta X^3_{,2} + X^2_{,3} = 0, \quad (11)$$

$$\frac{\dot{B}}{2 B} X^0 + \coth \theta X^2 + X^3_{,3} = 2 \delta. \quad (12)$$

Solving equations (7), (8) and (9) we get

$$X^0 = \delta t + P^1(\theta, \phi), X^2 = P^1_\theta(\theta, \phi) \int \frac{1}{B} dt + P^2(\theta, \phi), \quad (13)$$

$$X^3 = \frac{1}{\sinh^2 \theta} P^1_\phi(\theta, \phi) \int \frac{1}{B} dt + P^3(\theta, \phi),$$

where $P^1(\theta, \phi)$, $P^2(\theta, \phi)$ and $P^3(\theta, \phi)$ are functions of integration which are to be determined. To avoid details here we shall write only the results as follows:

$$X^0 = \delta t + d_1, X^1 = \alpha r + \beta, X^2 = d_2 \sin \phi + d_3 \cos \phi, \quad (14)$$

$$X^3 = \coth \theta (d_2 \cos \phi - d_3 \sin \phi) + d_4.$$

and the line element (5) now becomes

$$ds^2 = -dt^2 + dr^2 + (\delta t + d_1)^2 (d\theta^2 + \sinh^2 \theta d\phi^2). \quad (15)$$

In view of equations (14) and (15) we need to consider two distinct cases which are $\alpha = 0$ and $\alpha \neq 0$. If we choose $\alpha = 0$, $\delta \neq 0$ and subtract the Killing vector fields from (14) we will simply have

$$X^0 = \delta t + d_1, X^1 = X^2 = X^3 = 0 \quad (16)$$

here the self-similar vector field is non tilted and orthogonal to the fluid flow and represents the zero th kind self similarity. Now if $\alpha \neq 0$, $\delta = 0$ then after subtracting the Killing vector fields we have

$$X^0 = d_1, X^1 = \alpha r, X^2 = X^3 = 0 \quad (17)$$

with the line element

$$ds^2 = -dt^2 + dr^2 + d_1 (d\theta^2 + \sinh^2 \theta d\phi^2) \quad (18)$$

thus (17) shows that the self-similar vector field is parallel to the fluid flow and hence represents the infinite kind self similarity in non tilted parallel case. Finally if we put $\alpha \neq 0$ and $\delta \neq 0$ we have

$$X^0 = \delta t + d_1, X^1 = \alpha r, X^2 = X^3 = 0 \quad (19)$$

with line element

$$ds^2 = -dt^2 + dr^2 + (\delta t + d_1)^2 (d\theta^2 + \sinh^2 \theta d\phi^2) \quad (20)$$

here the vector field is tilted to the fluid flow and hence represents the self-similarity of second kind in the tilted case.

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حلول متشابهة ذاتية البناجي النمط الثالث للزمكان باستخدام المعادلات التفاضلية الجزئية

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الملخص

تم تطوير أساليب مختلفة لدراسة الحلول المتشابهة ذاتياً لبناجي النمط النهائي في الزمكان. وتم الحصول على الحلول المتشابهة ذاتياً لكلا الحالتين المغطاة وغير المغطاة. تم حل منظومة ست معادلات تفاضلية جزئية غير خطية وغير متجانسة. وضحت الحالة غير المغطاة أن الزمكان يسمح بنوع من التعامد الصفري متشابه ذاتياً، أما بالنسبة للحالة المغطاة فهي تقبل النوع الثاني واللا نهائي من التشابهات الذاتية.