Almahameed M., J. of the Association of Arab Univ. for Basic and Applied Science, Vol. 7, 2009, 66-75

Maximum Principles for Second Order Elliptic Systems

Mohammad Almahameed

Department of Mathematics, Irbid National University, Irbid, Jordan

ABSTRACT

In this paper we discuss a classical maximum principle for weakly coupled second order homogeneous elliptic systems. We find a sufficient condition for the classical maximum principle which extend the result of Winter and Wong for negative semidefinite matrix to a more general form.

1. INTRODUCTION

We first introduce some remarks and notations :

(1) Unless otherwise stated, all matrices considered in this paper will be over the complex field.

(2) Let A be an $m \times n$ matrix, then A^T, $\overline{A}$, and A^* $(A^* = \overline{A}^T)$ denote its transpose, complex conjugate, and adjoint, respectively.

(3) Both Hermitian positive definite and real symmetric positive definite matrices will be named positive. *Similar abbreviations hold for semipositive, negative, and seminegative definite matrices.*

(4) The notations $B > 0$, $B \geq 0$, $B < 0$, and $B \leq 0$ mean that the square matrix B is positive, semi-positive, negative and semi-negative, respectively.

(5) $||.||_{o,\Omega}$ denotes the sup norm over Ω; thus for complex valued vector functions $u = (u_1, u_2, ..., u_n)$. [Chow, Dunninger, (1974); Dow, (1975); Franciosi, (1989); Hille, Protter, 1977)]

$$||u||_{o,\Omega} = \sup_{x\in\Omega} |u(x)| = \sup_{x\in\Omega} (|u_1(x)|^2 + ... + |u_n(x)|^2)^{1/2}$$

In Liapunov's Second Method, positive definite solutions B of the matrix equation $A^*B + BA = -E$ $(E > 0)$ have been used to construct Liapunov functions, and to prove stability of some ordinary differential systems. $\frac{du}{dx} = Au$, [Massera, (1956); Protter and Weinberger, (1984);

Snyders and M. Zakai, (1970); Sperb, (1981); Staring, (1980); Takac, (1996); Wasowski, (1972); Winter, Wong, (1972); Yoshizaea, (1966)].

Chow and Dunninger (1974), used this method to obtain a generalized maximum principle for some classes of n-metaharmonic functions.

In this paper we use the idea of Liapunov's Second Method to find a generalized maximum principle for a class of weakly coupled second order homogeneous elliptic systems.

$$Lu + Au = 0 \quad \text{in} \quad \Omega \subset \boldsymbol{R}^n$$

Where L is the second order elliptic operator

$$L\,[u(x)] = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i}, \quad a_{ij} = a_{ji},\ u = (u_1, u_2, \ldots, u_n)^T.$$

The following Lemma is a well-known result in Liapunov stability theory and will be used in this paper.

Liapunov Lemma: Let A be an $n \times n$ complex or real matrix.

(a) Assume that no eigenvalues of A has positive real part, and moreover that the elementary divisors of A corresponding to eigenvalues with vanishing real part are linear. Then there exist matrices $B > 0$ and $E \geq 0$ such that.

$$A^* B + BA = -E$$

(b) If each eigenvalue of A has negative real part, then for any $E > 0$, there exist a unique $B > 0$ such that $A^*B + BA = -E$

The proof of this Lemma can be found in [Yoshizaea, (1966)].

2. THE GENERALIZED MAXIMUM PRINCIPLE

Consider a second order operator

$$L\,[u(x)] = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i}, \quad a_{ij} = a_{ji} \tag{2.1}$$

in a bounded domain Ω in $\mathbf{R}^n$. We assume that L is elliptic in Ω, i.e., for all $x \in \Omega$ and all $y=(y_1, y_2, \ldots, y_n) \in \boldsymbol{R}^n \setminus \{0\}$.

$$a_{ij}(x)\, y_i\, y_j > 0 \tag{2.2}$$

holds, we also suppose that the coefficients a_{ij} and a_i are bounded and real-valued functions in Ω. Now consider the following weakly coupled second order elliptic system,

$$Lu_s(x) + a_{sk}(x)\, u_k(x) = 0\ ,\ s = 1,2,\ \dots,\ n \quad in\ \ \Omega,$$

or in matrix from,

$$Lu(x) + A(x)\, u(x) = 0 \qquad in\ \ \Omega \qquad (2.3)$$

Here $A(x) = (a_{sk}(x))$ is an $n \times n$ complex matrix function and u is a $C^2\,[\,n\times 1]$complex vector function.

Associated with (2.3) the following characteristic equation of A,

$$|\lambda I - A| = 0.$$

Theorem (2.1): Assume that there exists a constant complex matrix B > 0 such that

$$A^*(x)\, B + BA(x) \leq 0, \quad x\in\Omega \qquad (2.4)$$

Then for all solutions $u\in C^2(\Omega)\cap C(\overline{\Omega})$ of (3.3) there exist a constant k > 0 such that

$$\|u\|_{o,\Omega} \leq k\ \ \|u\|_{o,\partial\Omega} \qquad (2.5)$$

Here $k = (\lambda_n / \lambda_1)^{1/2}$, where λ_1 and λ_n are the smallest and biggest eigenvalues of B, respectively.

Proof: Let

$$v = u^*Bu = u.\ Bu = Bu.u = b_{ks}\, \overline{u}_k\, u,$$

where "." denotes the dot product in $\boldsymbol{C}^n$ defined by $x.y = y.^*x = \sum_{k=1}^{n} x_k\, \overline{y}_k$.

Then v is a nonnegative function and,

$$v_i \equiv \frac{\partial v}{\partial x_i} = b_{ks}\, \overline{u}_{ki}\, u_s + b_{ks}\, \overline{u}_k\, u_{si},$$

$$v_{ij} \equiv \frac{\partial^2 v}{\partial x_i \partial x_j} = b_{ks}\, \overline{u}_{kij}\, u_s + b_{ks}\, \overline{u}_k\, u_{sij} + 2\ Re\ \{\, b_{ks}\, \overline{u}_k\, u_{sj}\},$$

where $u_{ki} \equiv \frac{\partial u_k}{\partial x_i}$, $u_{kij} \equiv \frac{\partial^2 v}{\partial x_i\, \partial x_j}$, etc, and,

$Lv = a_{ij}\, v_{ij} + a_i\, v_i$

$= b_{ks}\, a_{ij}\, \overline{u}_{kij}\, u_s + b_{ks}\, \overline{u}_k\, a_{ij}\, u_{sij} + 2\, b_{ks}\, a_{ij}\, \overline{u}_{ki}\, u_{sj}$

$+ b_{ks}\, a_i\, \overline{u}_{ki}\, u_s + b_{ks}\, \overline{u}_k\, a_i\, u_{si}$

$= b_{ks}\, \overline{(Lu_k)}\, u_s + b_{ks}\, \overline{u}_k\, (Lu_s) + 2\, a_{ij}\, u_i.\, B\, u_j = (Lu)^*\, Bu + u^*\, B\, (Lu) + 2\, a_{ij}\, B^{½}\, u_i.\, B^{½}\, u_j$

Thus $\quad Lv = -u^*\, (A^*B + BA)u + 2\, a_{ij}\, B^{½}\, u_i.\, B^{½}\, u_j \geq 0 \qquad (2.6)$

Since $A^*B + BA \leq 0$ and $a_{ij}\, v_i\, .\, v_j \geq 0$

For any vectors $v_1\, .\, v_2,\, \ldots,\, v_n$.

Therefore, by the maximum principle for the elliptic operator L, we have

$$v(x) \leq \max_{y \in \partial\Omega} v(y), \quad \forall x \in \Omega \qquad (2.7)$$

Suppose that

$\{\lambda i\}_{i=1}^{n}$ are the eigenvalues of B with $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$

Since $B > 0$, we know that

$$\lambda_1 > 0 \text{ and } \lambda_1\, |u(x)|^2 \leq v(x) = u(x)^*\, B\, u(x) \leq \lambda_n\, |u(x)|^2$$

Hence, from (2.7)

$$|u(x)|^2 \leq \frac{\lambda_n}{\lambda_1} \max_{y \in \partial\Omega} |u(y)|^2,$$

and $||u||_{0,\Omega} \leq k\, ||u||_{0,\partial\Omega}$ where $k = (\lambda_n/\lambda_1)^{½}$

The Liapunov lemma yields the following theorem.

Theorem (2.2): Let A(x) = g(x) I + D in (2.3), where g(x) ≤ 0 in Ω and D is a constant matrix over **C**. Assume that none of the eigenvalues of D has a positive real part, and moreover that the elementary divisors of D corresponding to eigenvalues with vanishing real part are linear. Then there exit a constant k > 0 such that for all solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of (2.3).

$$||u||_{0,\Omega} \leq k\, ||u||_{0,\partial\Omega}$$

Proof: by Liapunov lemma, there exist matrices $B > 0$ and $E \geq 0$ such that $D^*\, B + BD = -E \leq 0$.

Since $g \leq 0$ in Ω, we get

$$A^*(x)\, B + BA(x) = 2\, g(x)\, B + D^* B + BD \leq 0.$$

Now the result of this Theorem follows from Theorem (2.1).

Theorems (2.1) and (2.2) are generalized maximum principles since the value of *k* in (2.5) may be larger than 1.

The best value of *k* in (2.5) for any matrix *A*, is *k=1* which corresponds to the classical maximum principle.

3. THE CLASSICAL MAXIMUM PRINCIPLE

Theorem (3.1):

(a) A sufficient condition that:

$$\|u\|_{0,\Omega} \leq \|u\|_{0,\partial\Omega} \qquad (3.1)$$

holds , for all solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *of (2.3) is*

$$A^*(x) + A(x) \leq 0. \qquad (3.2)$$

(b) Assume that the variable matrix A=A(x) in (2.3) is normal (i.e., $A^*(x)\, A(x) = A(x)\, A^*(x)$, $x \in \Omega$), and all its eigenvalues have nonpositive real parts for all $x \in \Omega$.

Then (3.1) holds for all solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of (2.3) .

Proof. (a) By choosing *B=I* in Theorem (2.1), (2.5) with *K=1* (i.e., 3.1) follows from the condition (3.2)

(b) suppose

$$\lambda_1(x)\ \lambda_2(x), \ldots, \lambda_n(x)$$

are all the eigenvalues of *A(x)*. Since *A(x)* is normal, there exists a unitary matrix *U(x)* such that

$$U^*(x)\, A\,(x)\, U(x) = \begin{bmatrix} \lambda_1(x) & & & \\ & \lambda_2(x) & & \\ & & \ddots & \\ & & & \lambda_n(x) \end{bmatrix}$$

Therefore, by the assumption,

$$U^*(x)\,(A^*(x) + A(x))\, U(x) = \begin{bmatrix} 2\,Re\lambda_1(x) & & \\ & \ddots & \\ & & 2\,Re\,\lambda_n(x) \end{bmatrix} =: \Lambda(x) \le 0.$$

Hence $A^* + A = U\Lambda U^* \le 0;$ and then 3.1 follows from (a).

Example: For $n = 2$, consider

$$Lu + \begin{bmatrix} a & b \\ c & d \end{bmatrix} u = 0, \qquad a, b, c, d \in \mathbf{R}.$$

The associated characteristic equation,

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0,$$

has roots

$$\lambda_\pm = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$= \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$$

Hence, by Theorem (2.2) the inequality (2.5) is valid provided one of the following conditions is satisfied:

i. $a + d < 0$, $(a - d)^2 + 4bc \le 0$;

ii. $a + d < 0$, $(a - d)^2 + 4bc > 0$, $ad - bc \ge 0$;

iii. $a + d = 0$, $ad - bc > 0$.

The inequality (2.5) is not valid for the general case, when $a + d > 0$ or

$a + d = 0$, $ad - bc \leq 0$. In fact, $u = \sin x \sin y \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ solves the systems

$$\Delta u + \begin{bmatrix} 0 & -1 \\ 4 & 4 \end{bmatrix} u = 0$$

$$\Delta u + \begin{bmatrix} 0 & -1 \\ -4 & 0 \end{bmatrix} u = 0$$

in $\Omega = (0, \pi) \times (0, \pi)$ and vanishes on $\partial\Omega$ but (2.5) does not hold.

Theorem (2.1) gives a sufficient condition for which (2.5) holds. It raises some open questions as to whether Theorem (2.1) can be extended to a more general system (2.3) with weaker restrictions on the matrix A, and as to whether necessary conditions can be determined so that (2.5) holds.

Following from the inequality

$$L(u^* Bu) \geq -u^* (A^*B+BA)\, u + 2a_{ij}\, B^{½}u_i\, B^{½}\, u_j \geq 0, \qquad (2.6)$$

in the proof of Theorem (2.1), and form Protter and Weinberger [6], are the following two maximum principles for system (2.3)

Corollary (3.1): if $u\in C^2\,(\Omega)\cap C(\overline{\Omega})$ is a solution of (2.3), and if u^* Bu attains a maximum in Ω for some positive definite matrix B such that $A^*(x)$ B+BA(x) is negative semidefinite in Ω, then u is a complex constant vector in Ω. Moreover, if $A^*(x)$ B+ BA(x) is negative definite at some $x\in\Omega$ or, if A (x) is invertible for some $x\in\Omega$ then u= 0 in Ω .

Proof. Under the assumption of this corollary, by the proof of Theorem 2.1, inequality 2.6 holds. Thus, by the maximum principle of the second order elliptic equation (see [Sperb, (1981); Staring, (1980); Takac, (1996); Wasowski, (1972)]), u^*Bu = constant. Hence, from (2.6) again, we have

$$0 = L\,(u^* Bu)$$

$$= -u^*\,(A^*\,B+ BA)\,u + 2a_{ij}\,B^{½}\,u_i\,B^{½}\,u_j \geq 0 \qquad in\ \Omega$$

Which implies that $u^* (A^* B + BA) u = 0$ and $a_{ij} B^{½} u_i B^{½} u_j = 0$, and then.

$B^{½} u_i = 0$ and $u_i = 0$ in Ω for $1 \le i \le n$. Thus u is a complex constant vector in Ω, Moreover, if $A^*(x) B + BA(x) < 0$ at some $x \in \Omega$ then, from $u^*(A^*(x) B + BA(x) u = 0$, we have $u = 0$ in Ω and if A(x) is invertible for some $x \in \Omega$ then, from the system 2.3 we still have $u \equiv 0$ in Ω.

Corollary (3.2): Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of 2.3 Suppose that $u^* Bu \le M$ in Ω and that $u^* Bu = M$ at a point $P \in \partial \Omega$ for some positive definite B such that $A^* B + BA$ is negative semidefinite. Here M is a nonnegative constant. Assume that P lies on the boundary of a ball in Ω, and that the outward directional derivative $\partial u/\partial v$ exists at P. Then

$$\frac{\partial\left(u^* Bu\right)}{\partial v} = 2Re\left[u^* B\frac{\partial u}{\partial v}\right] = 2\ Re\left[\left(B^{1/2}u\right)^* \frac{\partial\left(B^{1/2}u\right)}{\partial v}\right] > 0 \qquad at\ P$$

Unless u is a complex constant vector such that $u^* Bu \equiv M$; equivalently,

$$\frac{\partial\left|B^{1/2}u\right|}{\partial v} > 0 \qquad\qquad at\ P$$

Unless u is constant and $|B^{½} u| = M^{½}$

4. CONCLUDING REMARKS

1. The condition 3.2 is also necessary for the proof of the classical maximum principle by the method imposed here.
2. Theorem 3.1 contains the result of [Winter and Wong, (1972)] for real negative semidefinite $A = A(x, u, \nabla u)$ as a special case; one may view, for given u, $A(x, u(x), \nabla u(x))$ as a matrix function $A_1(x)$.
3. By Liapunov Lemma, there exist at least one positive definite matrix $B > 0$, satisfying $D^* B + BD = -E \le 0$, if the matrix D meets the assumption of Theorem (2.2). See [Massera, (1956); Protter, Weinberger, (1984); Snyders, Zakai, (1970)].
4. Corollary 3.1 actually holds even if Ω is unbounded, since inequality (2.6) holds and if $A^*(x) B + BA(x) < 0$ at some $x \in \Omega$, then we still have $u = 0$ in Ω .

REFERENCES

Chow S. N. and Dunninger D. R., (1974) A Maximum Principle for n–Metaharmonic Function, *Proc. Amer, Math. Soc*, **43**, 79-83.

Dow M. A., (1975) Maximum principles for weakly coupled systems of quasilinear parabolic inequalities*, J. Austral. Math. Soc*. **19**, 79-83.

Franciosi M., (1989) Maximum Principles for Second Order Elliptic Equations and Applications , *J.Math.Anal.Appl*.(2)**138**, 343-348.

Hille G. N. and Protter M. H., (1977) Maximum Principles for a Class of First Order Elliptical Systems, J.Diff. Eqs. 24,136-151.

Massera, J. L. (1956) Contributions to Stability Theory, *Ann. Of Math*. (2) **64**, 182-206.

Protter M. H. and Weinberger H. F. (1984) Maximum Principles in Differential Equations Springer – verlag, Berlin,.

Snyders J. and Zakai M., (1970) On Nonnegative Solutions of the Equation $AD+DA^{*}= -C$, *SIAM J. Appl. Math*. **18**, 704-714.

Sperb R., (1981) Maximum Principles and Their Applications. Academic Press, New York,.

Staring, G. (1980) Linear Algebra and its Applications, Academic Press, New York,.

Takac P., (1996)An Abstract Form of Maximum and Anti-Maximum Principles of Hopfs Type , *J. Math. Anal. Appl*.(2)201 , 339-364.

Wasowski J., (1972) Maximum Principles for a certain strongly elliptic system of linear equations of second order, *Bull. Acad. Polon. Sci* **18**, 741-745.

Winter M. J. and Wong P., (1972) Comparison and maximum theorems for system of quasilinear elliptic differential equations, *J. Math. Anal. Appl*. **40**, 634-642.

Yoshizaea T., (1966) Stability Theory by Liapunov's Second Method, *The Mathematical Society of Japan*,.

Almahameed M., J. of the Association of Arab Univ. for Basic and Applied Science, Vol. 7, 2009, 66-75

مبادئ الحد الأعلى لمنظومة الرتبة الثانية الاهليليجية (الناقصية)

محمد المحاميد

قسم الرياضيات الجامعة الوطنية في اربد
إربد ، الأردن

الملخص

نناقش في هذه الورقة مبدأ الحد الأعلى الكلاسيكي لمنظومة الناقصية المتجانس ذات الرتبة الثانية للمتقارن الضعيف. نحصل على الشرط الكافي لمبدأ الحد الأعلى الكلاسيكي الذي من خلال تم توسعة نتيجة ونتر ودانج للمصفوفة السالبة شبه المحددة إلى صور أكثر عمومية.