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K-th Upper Record Values from Modified Weibull Distribution and Characterization

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Abstract: In this paper $k-$th upper record values from modified Weibull distribution has been studied and some new recurrence relations satisfied by single and product moments are derived. The results obtained are the generalization of those obtained by Sultan [9] and Balakrishnan and Chan [2]. Further, conditional expectation and recurrence relation for single moments are used to characterize the said distribution and some particular cases are also discussed.

Keywords: Order statistics, $k-$th upper record values, Single moments, Product moments, Recurrence relations, Modified Weibull distribution and characterization.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables with distribution function (df) $F(x)$ and probability density function (pdf) $f(x)$. The $j-$th order statistic of a sample $X_1, X_2, \ldots, X_n$ is denoted by $X_{j:n}$. For a fixed $k \geq 1$ we define the sequence $\{U_n^{(k)}, n \geq 1\}$ of $k-$th upper record times of $\{X_n, n \geq 1\}$ as follows:

$$U_1^{(k)} = 1$$

$$U_{n+1}^{(k)} = \min\{j > U_n^{(k)} : X_{j:j+k-1} > X_{U_n^{(k)}:U_n^{(k)}+k-1}\}.$$

For $k=1$ and $n=1,2,\ldots$, we write $U_n^{(1)} = U_n$. Then $\{U_n, n \geq 1\}$ is the sequence of record times of $\{X_n, n \geq 1\}$. The sequence $\{Y_n^{(k)}, n \geq 1\}$, where $Y_n^{(k)} = X_{U_n^{(k)}}$ is called the sequence of $k-$th upper record values of $\{X_n, n \geq 1\}$. For convenience, we shall also take $Y_0^{(k)} = 0$. Note that for $k=1$ we have $Y_n^{(1)} = X_{U_n}$, $n \geq 1$, which are the record values of $\{X_n, n \geq 1\}$ (Ahsanullah [1]).

Then the pdf of $Y_n^{(k)}$ and the joint pdf of $Y_m^{(k)}$ and $Y_n^{(k)}$ are as follows:

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{\Gamma(n)}[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1} f(x), \quad n \geq 1, \tag{1.1}$$

$$f_{Y_m^{(k)}, Y_n^{(k)}}(x,y) = \frac{k^n}{\Gamma m \Gamma(n-m)}[-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)}[\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1}[\bar{F}(y)]^{k-1} f(y),$$
$$x < y, \quad 1 \leq m < n, \quad n \geq 2, \tag{1.2}$$

where, $\bar{F}(x) = 1 - F(x)$.

(Dziubdziela and Kopociński [3], Grudzień [4]).

In this paper, we have established some simple recurrence relations for single and product moments of $k-$th upper record values from modified Weibull distribution. Further, various deductions and particular cases are discussed and

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two theorems for characterizing this distribution based on recurrence relation for single moments and conditional expectation of $k-$th upper record values are given.

A random variable X is said to have modified Weibull distribution (Lai *et al.* [7]) if its *pdf* is of the form

$$f(x)=\alpha(\beta+\lambda x)x^{\beta-1}e^{\lambda x}\exp(-\alpha x^{\beta}e^{\lambda x}),\quad x>0,\ \alpha>0,\ \beta\geq 0 \text{ and } \lambda>0 \tag{1.3}$$

with the corresponding *df*

$$F(x)=1-\exp(-\alpha x^{\beta}e^{\lambda x}),\quad x>0,\ \alpha>0,\ \beta\geq 0 \text{ and } \lambda>0. \tag{1.4}$$

It is easily observe that

$$x f(x)=(\beta+\lambda x)[-\ln\bar{F}(x)]\bar{F}(x). \tag{1.5}$$

Type I extreme value or log gamma or log Weibull and Weibull distributions are considered as special cases of this distribution when $\beta=0$ and $\lambda=0$, respectively. The Rayleigh distribution arises when $\beta=2$ and $\lambda=0$. The exponential distribution is obtained when $\beta=1$ and $\lambda=0$.

The relation in (1.5) will be exploited to derive some simple recurrence relations for the moments of $k-$th upper record values from the modified Weibull distribution and then used to characterize this distribution.

2. Relations for Single Moments

Theorem 2.1. Fix a positive integer $k\geq 1$, for $n\geq 1$, $n\geq k$ and $j=0,1,\ldots,$

$$E(Y_n^{(k)})^j=\frac{n\beta}{j}[E(Y_{n+1}^{(k)})^j-E(Y_n^{(k)})^j]+\frac{n\lambda}{j+1}[E(Y_{n+1}^{(k)})^{j+1}-E(Y_n^{(k)})^{j+1}]. \tag{2.1}$$

Proof. For $n\geq 1$ and $j=0,1,\ldots,$ we have from (1.1) and (1.5)

$$E(Y_n^{(k)})^j=\frac{k^n}{\Gamma n}\left\{\beta\int_0^{\infty}x^{j-1}[-\ln\bar{F}(x)]^n[\bar{F}(x)]^k dx+\lambda\int_0^{\infty}x^{j}[-\ln\bar{F}(x)]^n[\bar{F}(x)]^k dx\right\}$$

$$=(\beta I_{j-1}+\lambda I_j), \tag{2.2}$$

where

$$I_r=\frac{k^n}{\Gamma n}\int_0^{\infty}x^r[-\ln\bar{F}(x)]^n[\bar{F}(x)]^k dx.$$

Upon integrating by parts, we obtain

$$I_r=\frac{k^{n+1}}{(r+1)\Gamma n}\int_0^{\infty}x^{r+1}[-\ln\bar{F}(x)]^n[\bar{F}(x)]^{k-1}dx-\frac{nk^n}{(r+1)\Gamma n}\int_0^{\infty}x^{r+1}[-\ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}dx$$

$$=\frac{n}{r+1}[E(Y_{n+1}^{(k)})^{r+1}-E(Y_n^{(k)})^{r+1}].$$

Substituting for I_j and I_{j-1} in (2.2) and simplifying the resulting expression, we derive the relation given in (2.1).

Remark 2.1.

i) Setting $\beta=0$ in (2.1), we get the result for single moments of $k-$th upper record values obtained by Selim and Salem [8] for type I extreme value distribution or log gamma or log Weibull distribution.

ii) Putting $\lambda=0$ in (2.1), we deduce the recurrence relation for single moments of $k-$th upper record values for Weibull distribution in the form

$$E(Y_{n+1}^{(k)})^j=\left(\frac{j+n\beta}{n\beta}\right)E(Y_n^{(k)})^j.$$

iii) Putting $\lambda=0$ and $\beta=1$ in (2.1), the recurrence relation for single moments of $k-$th upper record values from exponential distribution can be obtained as

$$E(Y_{n+1}^{(k)})^j=\left(\frac{j+n}{n}\right)E(Y_n^{(k)})^j.$$

iv) Setting $\lambda = 0$ and $\beta = 2$ in (2.1), we obtain a recurrence relation for single moments of $k-$th upper record values for Rayleigh distribution in the form

$$E(Y_{n+1}^{(k)})^j = \left(\frac{j+2n}{2n}\right) E(Y_n^{(k)})^j .$$

Corollary 2.1. The recurrence relation for single moments of upper record values from the modified Weibull distribution has the form

$$E(X_{U_n}^j) = \frac{n\beta}{j}\{E(X_{U_{n+1}}^j) - E(X_{U_n}^j)\} + \frac{n\lambda}{j+1}\{E(X_{U_{n+1}}^{j+1}) - E(X_{U_n}^{j+1})\} . \tag{2.3}$$

The above relation was also shown by Sultan [9].

Remark 2.2.

i) Setting $\beta = 0$ in (2.3), we get the recurrence relation for single moments of upper record values from type I extreme value distribution as obtained by Selim and Salem [8].

ii) If $\lambda = 0$ in (2.3), the result for single moments of upper record values obtained by Balakrishnan and Chan [2] for Weibull distribution is deduced.

iii) If $\lambda = 0$ and $\beta = 1$ in (2.3), the result for single moments of upper record values is deduced for exponential distribution, which verify the result obtained by Balakrishnan and Chan [2].

iv) If $\lambda = 0$ and $\beta = 2$ in (2.3), the result for single moments of upper record values is deduced for Rayleigh distribution as obtained by Balakrishnan and Chan [2].

3. Relations for Product Moments

Theorem 3.1. For $m \geq 1$, $m \geq k$ and $i, j = 0,1,\ldots$,

$$E[(Y_m^{(k)})^i (Y_{m+1}^{(k)})^j] = \frac{\beta m}{i}\{E(Y_{m+1}^{(k)})^{i+j} - E[(Y_m^{(k)})^i (Y_{m+1}^{(k)})^j]\} + \frac{\lambda m}{i+1}\{E(Y_{m+1}^{(k)})^{i+j+1} - E[(Y_m^{(k)})^{i+1} (Y_{m+1}^{(k)})^j]\} . \tag{3.1}$$

and for $1 \leq m \leq n-2$, $i, j = 0,1,\ldots$,

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = \frac{\beta m}{i}\{E[(Y_{m+1}^{(k)})^i (Y_n^{(k)})^j] - E[(Y_m^{(k)})^i (Y_n^{(k)})^j]\}$$

$$+ \frac{\lambda m}{i+1}\{E[(Y_{m+1}^{(k)})^{i+1} (Y_n^{(k)})^j] - E[(Y_m^{(k)})^{i+1} (Y_n^{(k)})^j]\} . \tag{3.2}$$

Proof. From (1.2), for $1 \leq m \leq n-1$ and $i, j = 0,1,\ldots$, we have

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = \frac{k^n}{\Gamma(m)\Gamma(n-m)} \int_0^\infty y^j [\bar{F}(y)]^{k-1} f(y) I(y) dy , \tag{3.3}$$

where

$$I(y) = \int_0^y x^i [-\ln \bar{F}(x)]^{m-1} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} \frac{f(x)}{\bar{F}(x)} dx .$$

Integrating $I(y)$ by parts and using (1.5), we obtain

$$I(y) = \frac{\beta(n-m-1)}{i} \int_0^y x^i [-\ln \bar{F}(x)]^m [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-2} \frac{f(x)}{\bar{F}(x)} dx$$

$$- \frac{\beta m}{i} \int_0^y x^i [-\ln \bar{F}(x)]^{m-1} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} \frac{f(x)}{\bar{F}(x)} dx$$

$$+ \frac{\lambda(n-m-1)}{i+1} \int_0^y x^{i+1} [-\ln \bar{F}(x)]^m [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-2} \frac{f(x)}{\bar{F}(x)} dx$$

$$- \frac{\lambda m}{i+1} \int_0^y x^{i+1} [-\ln \bar{F}(x)]^{m-1} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} \frac{f(x)}{\bar{F}(x)} dx .$$

Substituting this expression into (3.3) and simplifying, it leads to (3.2). Proceeding in a similar manner for the case $n = m+1$, the recurrence relation given in (3.1) can easily be established.

For $j = 0$, the result for product moments in (3.2) reduces to the relation for single moments as given in (2.1).

Remark 3.1.

i) Setting $\beta = 0$ in (3.2), the relation for the product moments in Selim and Salem [8] for $k-$th upper record values from the type I extreme value distribution is deduced.

ii) Putting $\lambda = 0$ in (3.2), we obtain the recurrence relation for the product moments of $k-$th upper record values for Weibull distribution in the form

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = \frac{\beta m}{i}\{E[(Y_{m+1}^{(k)})^i (Y_n^{(k)})^j] - E[(Y_m^{(k)})^i (Y_n^{(k)})^j]\}.$$

iii) Setting $\lambda = 0$ and $\beta = 1$ in (3.2), we get the recurrence relation for the product moments of $k-$th upper record values from exponential distribution has the form

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = \frac{m}{i}\{E[(Y_{m+1}^{(k)})^i (Y_n^{(k)})^j] - E[(Y_m^{(k)})^i (Y_n^{(k)})^j]\}.$$

iv) Setting $\lambda = 0$ and $\beta = 2$ in (3.2), the result for product moments of $k-$th upper record values is deduced for Rayleigh distribution as

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = \frac{2m}{i}\{E[(Y_{m+1}^{(k)})^i (Y_n^{(k)})^j] - E[(Y_m^{(k)})^i (Y_n^{(k)})^j]\}.$$

Corollary 3.1. The recurrence relation for the product moments of upper record values from the modified Weibull distribution has the form

$$E(X_{U_m}^i X_{U_n}^j) = \frac{\beta m}{i}\{E(X_{U_{m+1}}^i X_{U_n}^j) - E(X_{U_m}^i X_{U_n}^j)\} + \frac{\lambda m}{i+1}\{E(X_{U_{m+1}}^{i+1} X_{U_n}^j) - E(X_{U_m}^{i+1} X_{U_n}^j)\}. \tag{3.4}$$

A similar result was obtained by Sultan [9].

Remark 3.2.

i) Setting $\beta = 0$ in (3.4), the result for the product moments of upper record values is deduced for type I extreme value distribution, established by Selim and Salem [8].

ii) If $\lambda = 0$ in (3.4), the result for the product moments of upper record values as obtained by Balakrishnan and Chan [2] for Weibull distribution is deduced.

iii) If $\lambda = 0$ and $\beta = 1$ in (3.4), the result for the product moments of upper record values is deduced for exponential distribution, which verify the result obtained by Balakrishnan and Chan [2].

iv) If $\lambda = 0$ and $\beta = 2$ in (3.4), the result for the product moments of upper record values is deduced for Rayleigh distribution as obtained by Balakrishnan and Chan [2].

4. CHARACTERIZATIONS

Theorem 4.1. For a positive integer $k \geq 1$ and j be a non-negative integer, a necessary and sufficient condition for a random variable X to be distributed with pdf given by (1.3) is that

$$E(Y_n^{(k)})^j = \frac{n\beta}{j}[E(Y_{n+1}^{(k)})^j - E(Y_n^{(k)})^j] + \frac{n\lambda}{j+1}[E(Y_{n+1}^{(k)})^{j+1} - E(Y_n^{(k)})^{j+1}] \tag{4.1}$$

for $n = 1, 2, \ldots,\ n \geq k$.

Proof. The necessary part follows from (2.1). On the other hand if the recurrence relation in (4.1) is satisfied, then on rearranging the terms in (4.1) and using (1.1), we have

$$\frac{k^n}{\Gamma n}\int_0^\infty x^j[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1} f(x)dx = \frac{\beta k^{n+1}}{j\,\Gamma n}\int_0^\infty x^j[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1} f(x)\left\{-\ln \bar{F}(x)-\frac{n}{k}\right\}dx$$
$$+\frac{\lambda k^{n+1}}{(j+1)\Gamma n}\int_0^\infty x^{j+1}[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1} f(x)\left\{-\ln \bar{F}(x)-\frac{n}{k}\right\}dx. \quad (4.2)$$

Let

$$h(x) = -\frac{1}{k}[-\ln \bar{F}(x)]^n[\bar{F}(x)]^k. \quad (4.3)$$

Differentiating both the sides of (4.3), we get

$$h'(x) = [-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1} f(x)\left\{-\ln \bar{F}(x)-\frac{n}{k}\right\}.$$

Thus

$$\frac{k^n}{\Gamma n}\int_0^\infty x^j[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1} f(x)dx = \frac{\beta k^{n+1}}{j\,\Gamma n}\int_0^\infty x^j h'(x)dx + \frac{\lambda k^{n+1}}{(j+1)\Gamma n}\int_0^\infty x^{j+1}h'(x)dx. \quad (4.4)$$

Integrating right hand side in (4.4) by parts and using the value of $h(x)$ from (4.3), we find that

$$\frac{k^n}{\Gamma n}\int_0^\infty x^j[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}\left\{f(x)-\left(\frac{\beta}{x}+\lambda\right)[-\ln \bar{F}(x)]\,\bar{F}(x)\right\}dx = 0. \quad (4.5)$$

Applying now a generalization of the Müntz-Szász Theorem (see for example Hwang and Lin [6]) to (4.5), we have

$$x\,f(x) = (\beta+\lambda x)[-\ln \bar{F}(x)]\bar{F}(x)$$

which proves that

$$F(x) = 1-\exp(-\alpha x^\beta e^{\lambda x}), \quad x>0,\ \alpha>0,\ \beta\ge 0 \text{ and } \lambda>0.$$

Theorem 4.2. Let X be a non-negative random variable having an absolutely continuous df $F(x)$ with $F(0)=0$ and $0\le F(x)\le 1$ for all $x>0$, then

$$E[\xi(Y_n^{(k)})\,|\,(Y_l^{(k)})=x] = \exp(-\alpha x^\beta e^{\lambda x})\left(\frac{k}{k+1}\right)^{n-l}, \qquad l=m, m+1,\ m\ge k \quad (4.6)$$

if and only if

$$\bar{F}(x) = \exp(-\alpha x^\beta e^{\lambda x}), \quad x>0,\ \alpha>0,\ \beta\ge 0 \text{ and } \lambda>0,$$

where

$$\xi(y) = \exp(-\alpha y^\beta e^{\lambda y}).$$

Proof. We have from (1.2) and (1.1)

$$E[\xi(Y_n^{(k)})\,|\,(Y_m^{(k)})=x] = \frac{k^{n-m}}{\Gamma(n-m)}\int_x^\infty \exp(-\alpha y^\beta e^{\lambda y})[\ln \bar{F}(x)-\ln \bar{F}(y)]^{n-m-1}\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{k-1}\frac{f(y)}{\bar{F}(x)}dy. \quad (4.7)$$

By setting $u=\dfrac{\bar{F}(y)}{\bar{F}(x)}=\dfrac{\exp(-\alpha y^\beta e^{\lambda y})}{\exp(-\alpha x^\beta e^{\lambda x})}$ from (1.4) in (4.7), we get

$$E[\xi(Y_n^{(k)})\,|\,(Y_m^{(k)})=x] = \frac{k^{n-m}}{\Gamma(n-m)}\exp(-\alpha x^\beta e^{\lambda x})\int_0^1 u^k(-\ln u)^{n-m-1}du. \quad (4.8)$$

We have Gradshteyn and Ryzhik [5] pp. 551

$$\int_0^1(-\ln x)^{\mu-1}x^{\upsilon-1}dx = \frac{\Gamma\mu}{\upsilon^\mu}, \quad \mu>0, \quad \upsilon>0. \quad (4.9)$$

On using (4.9) in (4.8), we can obtain the result given in (4.6).

To prove sufficient part, we have

$$\frac{k^{n-m}}{\Gamma(n-m)}\int_x^\infty \exp(-\alpha\, y^\beta e^{\lambda y})[\ln \bar{F}(x)-\ln \bar{F}(y)]^{n-m-1}[\bar{F}(y)]^{k-1} f(y)dy = [\bar{F}(x)]^k g_{n|m}(x)\,, \qquad (4.10)$$

where

$$g_{n|m}(x)=\exp(-\alpha\, x^\beta e^{\lambda x})\left(\frac{k}{k+1}\right)^{n-m}.$$

Differentiating (4.10) both the sides with respect to x, we get

$$-\frac{k^{n-m} f(x)}{\bar{F}(x)\Gamma(n-m-1)}\int_x^\infty \exp(-\alpha\, y^\beta e^{\lambda y})[\ln \bar{F}(x)-\ln \bar{F}(y)]^{n-m-2}$$
$$\times[\bar{F}(y)]^{k-1} f(y)dy = g'_{n|m}(x)[\bar{F}(x)]^k - k\, g_{n|m}(x)[\bar{F}(x)]^{k-1} f(x)$$

or

$$-k\, g_{n|m+1}(x)[\bar{F}(x)]^{k-1} f(x) = g'_{n|m}(x)[\bar{F}(x)]^k - k\, g_{n|m}(x)[\bar{F}(x)]^{k-1} f(x)\,.$$

Therefore,

$$\frac{f(x)}{\bar{F}(x)} = -\frac{g'_{n|m}(x)}{k[g_{n|m+1}(x)-g_{n|m}(x)]} = \alpha x^{\beta-1}e^{\lambda x}(\beta+\lambda x)\,, \qquad (4.11)$$

where

$$g'_{n|m}(x) = -\alpha e^{\lambda x}x^{\beta-1}(\beta+\lambda x)\exp(-\alpha\, x^\beta e^{\lambda x})\left(\frac{k}{k+1}\right)^{n-m}$$

$$g_{n|m+1}(x)-g_{n|m}(x)=\frac{1}{k}\exp(-\alpha\, x^\beta e^{\lambda x})\left(\frac{k}{k+1}\right)^{n-m}.$$

Integrating both sides of (4.11) with respect to x between $(0, y)$, the sufficiency part is proved.

Remark 4.1. Theorems 4.1, 4.2 can be used to characterize the type I extreme value, Weibull, exponential and Rayleigh distributions by setting parameters.

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