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ORIGINAL ARTICLE

On the limitations of linear growth rates in triply diffusive convection in porous medium

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Abstract The present paper purports to deal with the problem of triply diffusive convection in sparsely distributed porous medium using the Darcy-Brinkman model. Bounds are derived for the modulus of the complex growth rate p of an arbitrary oscillatory perturbation of growing amplitude, neutral or unstable for this configuration of relevance in oceanography, geophysics as well as in many engineering applications. These bounds are obtained by deriving the integral estimates for the various physical quantities by exploiting the coupling between them in the governing equations; and are important especially when at least one boundary is rigid so that exact solutions in the closed form are not obtainable. It is further proved that the result obtain herein is uniformly valid for any combination of rigid and dynamically free boundaries.

1. Introduction

Research on convective fluid motion in porous media under the simultaneous action of a uniform vertical temperature gradient and a gravitationally opposite uniform vertical concentration gradient (known as double diffusive convection) has been an area of great activity due to its importance in the prediction of ground water movement in aquifers, in assessing the effectiveness of fibrous materials, in engineering geology and in nuclear engineering. Double diffusive convection is now well known. For a broad view of the subject one may refer to Vafai (2005), Nield and Bezan (2006), Murray and Chen (1989), Chamkha et al. (2002), Umavathi et al. (2005), Zeeshan and Ellahi (2013), Ellahi et al. (2013, 2015), Rashidi et al. (2014, 2015), Hassan and Rashidi (2014), Zeeshan et al. (2014).

Although the double-diffusive convection in porous and nonporous medium is still an active research domain (Swamy, 2014; Choudhary et al., 2016; Khan and Sultan, 2015; Nield et al., 2015; Slim, 2014; Yang et al., 2015; Babu et al., 2014; Chamkha and Al-Naser, 2001; Magyari and Chamkha, 2008; Chamkha et al., 2010; Ellahi et al., 2012; Sheikholeslami et al., 2014a,b; Sheikholeslami and Ellahi, 2015), there are many physical configurations in which more than two diffusing components are present. For example Degens et al. (1973) reported that the saline waters of geothermally heated Lake Kivu are strongly stratified by temperature and salinity which is the sum of comparable concentrations of many salts, while the sea-water contains many salts in concentrations slightly less than the sodium chloride concentration.

The subject of systems having more than two components in porous and nonporous medium has attracted many

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researchers (Griffiths, 1979a; Poulikakos, 1985; Rudraiah and Vortmeyer, 1982; Lopez et al., 1990; Tracey, 1998; Prakash et al., 2015a,b; Ryzhkov and Shevtsova, 2009; Rionero, 2013; Zhao et al., 2014) due to its importance in the study of geothermally heated lakes, earth core, solidification of molten alloys, underground water flow, acid rain effects, natural phenomena such as contaminant transport, warming of stratosphere and magmas and their laboratory models and sea water etc. Some fundamental differences between the double and triply diffusive convection are noticed by these researchers. Among these differences one is that if the gradients of two of the stratifying agencies are held fixed, then three critical values of the Rayleigh number of the third agency are sometimes required to specify the linear stability criteria (in double diffusive convection only one critical Rayleigh number is required). Another is that the onset of convection may occur via a quassi periodic bifurcation from the motionless basic state.

The presence of more than two components in a fluid, each influencing the density and having different diffusive properties, can lead to convective instabilities, often well before a fluid system would become statically unstable. It is now well established that (Griffiths, 1979a,b; Terrones, 1993) the small concentration of a third component with a smaller mass diffusivity can have a significant effect upon the nature of instability; and 'diffusive convection' (oscillatory modes) and direct 'salt finger' modes (steady convection) may simultaneously exist under a wide range of conditions, even if the over-all density stratification is gravitationally stable. Thus, since instability in triply diffusive configuration may occur in the form of oscillatory motions, the problem of deriving the upper limits for the linear growth rate of an arbitrary neutral or unstable oscillatory disturbance of growing amplitude in triply diffusive convection has its own importance in fluid dynamics, especially when at least one of the boundaries is rigid so that exact solutions in the closed form are not derivable as was possible for the cases treated by Griffiths (1979a), Poulikakos (1985) and Rudraiah and Vortmeyer (1982). Banerjee et al. (1981) formulated a novel way of combining the governing equations and the boundary conditions for double diffusive convection problem so that a semicircle theorem is derivable and which in turn yields the desired bounds. Their method has been used to derive the desired bounds for triply diffusive convection in porous medium. Further the result for double diffusive convection in porous medium also follows as a consequence.

In the present paper we have studied triply diffusive convection in a sparsely distributed porous medium by using Darcy-Brinkman model. Darcy flow model is relevant only to densely packed, low permeability porous medium. Darcy's law cannot account for the no-slip boundary condition at the interface of a porous medium and a solid boundary (Kaviany, 1995). Also, experiments conducted with several combinations of solids and fluids covering wide ranges of governing parameters indicate that most of the experimental data do not agree with the theoretical predictions based on the Darcy flow model. The Brinkman (1947) extension of the Darcy's law gets around the obstacle by adding a viscous like term to the equations. Givler and Altobelli (1994) have demonstrated that for high permeability porous media the effective viscosity is about ten times the fluid viscosity. Therefore, the effect of viscosities on the stability analysis is of practical interest.

2. Mathematical formulation

A viscous finitely heat conducting Boussinesq fluid layer, saturating a porous medium, of infinite horizontal extension is statically confined between two horizontal boundaries $z = 0$ and $z = d$ which are respectively maintained at uniform temperatures T_0 and $T_1 (< T_0)$ and uniform concentrations S_{10}, S_{20} and S_{11} $(< S_{10}), S_{21}$ $(< S_{20})$ (as shown in Fig. 1). It is assumed that the saturating fluid and the porous layer are incompressible and that the porous medium is a constant porosity medium. It is further assumed that the cross-diffusion effects of the stratifying agencies can be neglected. The Brinkman extended Darcy model has been used to investigate the triple diffusive convection in porous medium.

The basic hydrodynamic equations that govern the problem are given by Vafai (2005).

Equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{1}$$

Equations of motion

$$\frac{1}{\epsilon}\frac{\partial u}{\partial t} + \frac{1}{\epsilon^2}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial}{\partial x}\left(\frac{\check{p}}{\rho_0}\right) - \frac{v}{k_1}u + \frac{\mu_e}{\rho_0}\nabla^2 u, \tag{2}$$

$$\frac{1}{\epsilon}\frac{\partial v}{\partial t} + \frac{1}{\epsilon^2}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial}{\partial y}\left(\frac{\check{p}}{\rho_0}\right) - \frac{v}{k_1}v + \frac{\mu_e}{\rho_0}\nabla^2 v, \tag{3}$$

$$\frac{1}{\epsilon}\frac{\partial w}{\partial t} + \frac{1}{\epsilon^2}\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial}{\partial z}\left(\frac{\check{p}}{\rho_0}\right) - \frac{v}{k_1}w + \frac{\mu_e}{\rho_0}\nabla^2 w - \frac{\rho}{\rho_0}g. \tag{4}$$

Equation of heat conduction

$$E\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \kappa\nabla^2 T. \tag{5}$$

Equation of mass diffusion

$$E_1\frac{\partial S_1}{\partial t} + u\frac{\partial S_1}{\partial x} + v\frac{\partial S_1}{\partial y} + w\frac{\partial S_1}{\partial z} = \kappa_1\nabla^2 S_1, \tag{6}$$

$$E_2\frac{\partial S_2}{\partial t} + u\frac{\partial S_2}{\partial x} + v\frac{\partial S_2}{\partial y} + w\frac{\partial S_2}{\partial z} = \kappa_2\nabla^2 S_2. \tag{7}$$

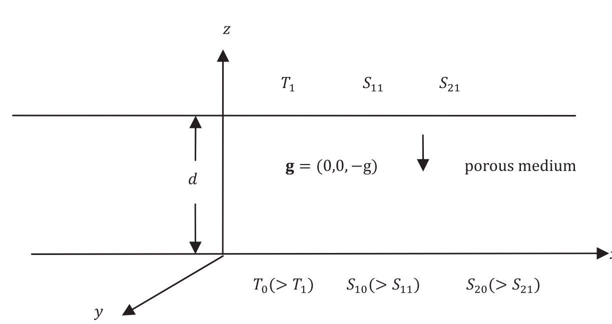

Figure 1 Physical configuration.

Equation of state

$$\rho = \rho_0[1 + \alpha(T_0 - T) - \alpha_1(S_{10} - S_1) - \alpha_2(S_{20} - S_2)], \tag{8}$$

where $\delta\rho = -\rho_0\alpha(T - T_0)$, (9)

$$\delta\rho' = \rho_0\alpha_1(S_1 - S_{10}), \tag{10}$$

$$\delta\rho'' = \rho_0\alpha_2(S_2 - S_{20}), \tag{11}$$

where u, v, w are the components of velocity in the x, y, z-directions respectively, and $\frac{\check{p}}{\rho_0}$ is the modified hydrodynamic pressure. Further $t, \rho, T, S_1, S_2, \epsilon, k_1, \mu_e, v, \kappa, \kappa_1$ and κ_2 are, respectively, the time, the density, the temperature, the concentration of first component, the concentration of second component, the porosity of the porous medium, the permeability of the porous medium, the effective viscosity, the kinematic viscosity, the thermal diffusivity, mass diffusivity of first component and the mass diffusivity of second component; α, α_1 and α_2 are respectively the coefficients of volume expansion due to temperature variation, concentration variation of first component and concentration variation of second component. Here $E = \epsilon + (1 - \epsilon)\frac{\rho_s C_s}{\rho_0 C_f}$ is a constant and E_1 and E_2 is also a constant analogous to E but corresponding to concentration rather than heat, where ρ_s, C_s and ρ_0, C_f stand for density and heat capacity of the solid (porous matrix) material and fluid respectively. The suffix '0' denotes the values of the various parameters at some suitably chosen reference temperature T_0 and concentration S_0.

The basic state is assumed to be quiescent state and is given by

$$(u, v, w) \equiv (0, 0, 0), \quad p \equiv p(z), \quad T \equiv T(z),$$
$$S_1 \equiv S_1(z), \quad S_2 \equiv S_2(z), \quad \rho \equiv \rho(z). \tag{12}$$

Thus the initial stationary state solutions are given by

$$(u,v,w) = (0,0,0), \quad \frac{\check{p}}{\rho_0} = P = P_0 - g\rho_0\left[z + (\alpha\beta - \alpha_1\beta_1 - \alpha_2\beta_2)\frac{z^2}{2}\right],$$
$$T = T_0 - \beta z, \quad S_1 = S_{10} - \beta_1 z, \quad S_2 = S_{20} - \beta_2 z,$$
$$\rho = \rho_0[1 + (\alpha\beta - \alpha_1\beta_1 - \alpha_2\beta_2)z], \tag{13}$$

where P_0 represents the pressure at the lower boundary $z = 0$, $\beta = \frac{T_0 - T_1}{d}$ is the maintained uniform adverse temperature gradient, $\beta_1 = \frac{S_{10} - S_{11}}{d}$ and $\beta_2 = \frac{S_{20} - S_{21}}{d}$ are the non-adverse concentration gradients. To study the stability of the system, we perturb all the variables in the form

$$\left.\begin{aligned}
&(\bar{u}, \bar{v}, \bar{w}) = (0 + u', 0 + v', 0 + w'),\\
&\check{p} = P_0 - g\rho_0\left[z + (\alpha\beta - \alpha_1\beta_1 - \alpha_2\beta_2)\frac{z^2}{2}\right] + \delta P',\\
&\bar{T} = T_0 - \beta z + \theta',\\
&\bar{S}_1 = S_{10} - \beta_1 z + \phi'_1,\\
&\bar{S}_2 = S_{20} - \beta_2 z + \phi'_2,\\
&\bar{\rho} = \rho_0[1 + \alpha(T_0 - T - \theta') - \alpha_1(S_{10} - S_1 - \phi'_1) - \alpha_2(S_{20} - S_2 + \phi'_2)],
\end{aligned}\right\} \tag{14}$$

where u', v', w', θ', ϕ'_1, ϕ'_2 and $\delta P'$ are the perturbed variables and assumed to be small.

Substituting Eq. (14) into Eqs. (1)–(7), we obtain the following linearized perturbation equations

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \tag{15}$$

$$\frac{1}{\epsilon}\frac{\partial u'}{\partial t} = -\frac{\partial(\delta P')}{\partial x} - \frac{v}{k_1}u' + \frac{\mu_e}{\rho_0}\nabla^2 u', \tag{16}$$

$$\frac{1}{\epsilon}\frac{\partial v'}{\partial t} = -\frac{\partial(\delta P')}{\partial y} - \frac{v}{k_1}v' + \frac{\mu_e}{\rho_0}\nabla^2 v', \tag{17}$$

$$\frac{1}{\epsilon}\frac{\partial w'}{\partial t} = -\frac{\partial(\delta P')}{\partial z} - \frac{v}{k_1}w' + \frac{\mu_e}{\rho_0}\nabla^2 w' + g\alpha\theta' - g\alpha_1\phi'_1 - g\alpha_2\phi'_2, \tag{18}$$

$$E\frac{\partial\theta'}{\partial t} - \beta w' = \kappa\nabla^2\theta', \tag{19}$$

$$E_1\frac{\partial\phi'_1}{\partial t} - \beta_1 w' = \kappa_1\nabla^2\phi'_1, \tag{20}$$

and $E_2\frac{\partial\phi'_2}{\partial t} - \beta_2 w' = \kappa_2\nabla^2\phi'_2.$ (21)

The normal mode expansion of the dependent variables u', v', w', θ', ϕ'_1, ϕ'_2 and $\delta P'$ is assumed in the form

$$F'(x, y, z, t) = F''(z)\exp[i(k_x x + k_y y) + nt], \tag{22}$$

where $k = \sqrt{(k_x^2 + k_y^2)}$ is the wave number of perturbation, k_x and k_y being real constants and n is a constant which can be complex in general.

For functions with these dependences on x, y and t, we have

$$\frac{\partial}{\partial t} = n, \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = -k^2 \text{ and } \nabla^2 = \frac{d^2}{dz^2} - k^2 \tag{23}$$

Eqs. (15)–(21), thus becomes

$$ik_x u'' + ik_y v'' + \frac{dw''}{dz} = 0, \tag{24}$$

$$\frac{1}{\epsilon}nu'' = -ik_x(\delta P'') - \frac{v}{k_1}u'' + \frac{\mu_e}{\rho_0}\left(\frac{d^2}{dz^2} - k^2\right)u'', \tag{25}$$

$$\frac{1}{\epsilon}nv'' = -ik_y(\delta P'') - \frac{v}{k_1}v'' + \frac{\mu_e}{\rho_0}\left(\frac{d^2}{dz^2} - k^2\right)v'', \tag{26}$$

$$\frac{1}{\epsilon}nw'' = -\frac{d(\delta P'')}{dz} - \frac{v}{k_1}w'' + \frac{\mu_e}{\rho_0}\left(\frac{d^2}{dz^2} - k^2\right)w'' + g\alpha\theta'' - g\alpha_1\phi''_1 - g\alpha_2\phi''_2, \tag{27}$$

$$En\theta'' - \beta w'' = \kappa\left(\frac{d^2}{dz^2} - k^2\right)\theta'', \tag{28}$$

$$E_1 n\phi'' - \beta_1 w'' = \kappa_1\left(\frac{d^2}{dz^2} - k^2\right)\phi''_1, \tag{29}$$

$$E_2 n\phi'' - \beta_2 w'' = \kappa_2\left(\frac{d^2}{dz^2} - k^2\right)\phi''_2. \tag{30}$$

Eliminating u'' and v'' from Eqs. (25) and (26) by multiplying Eq. (25) by ik_x and (26) by ik_y respectively, adding the resulting equations and using Eq. (24) and then eliminating $\delta P''$ between this resulting equation and Eq. (27), we obtain

$$\frac{\mu_e}{\rho_0}\left(\frac{d^2}{dz^2}-k^2\right)^2 w'' - \left(\frac{n}{\epsilon}+\frac{\nu}{k_1}\right)\left(\frac{d^2}{dz^2}-k^2\right)w''$$
$$= k^2\left(g\alpha\theta'' - g\alpha_1\phi_1'' - g\alpha_2\phi_2''\right). \quad (31)$$

Also Eqs. (28)–(30) can be written as

$$\left(\frac{d^2}{dz^2}-k^2-\frac{En}{\kappa}\right)\theta'' = -\frac{\beta w''}{\kappa}, \quad (32)$$

$$\left(\frac{d^2}{dz^2}-k^2-\frac{E_1 n}{\kappa_1}\right)\phi_1'' = -\frac{\beta_1 w''}{\kappa_1}, \quad (33)$$

$$\left(\frac{d^2}{dz^2}-k^2-\frac{E_2 n}{\kappa_2}\right)\phi_2'' = -\frac{\beta_2 w''}{\kappa_2}. \quad (34)$$

Now using the following non-dimensional parameters

$$a_* = kd, \quad z_* = \frac{z}{d}, \quad \tau_{1*} = \frac{\kappa_1}{\kappa}, \quad \tau_{2*} = \frac{\kappa_2}{\kappa}, \quad \sigma_* = \frac{\nu\epsilon}{\kappa},$$

$$D_a = \frac{k_1}{d^2}, \quad D_* = d\frac{d}{dz}, \quad p_* = \frac{nd^2}{\nu\epsilon}, \quad w_* = \frac{d}{\nu}w'',$$

$$R_* = \frac{g\alpha\beta d^4}{\kappa\nu}, \quad R_{1*} = \frac{g\alpha_1\beta_1 d^4}{\kappa\nu}, \quad R_{2*} = \frac{g\alpha_2\beta_2 d^4}{\kappa\nu},$$

$$\theta_* = \frac{\kappa}{\beta\nu d}\theta'', \quad \phi_{1*} = \frac{\kappa}{\beta_1\nu d}\phi_1'', \quad \phi_{2*} = \frac{\kappa}{\beta_{21}\nu d}\phi_2'', \quad \Lambda = \frac{\mu_e}{\mu}, \quad (35)$$

we can reduce Eqs. (31)–(34) to the following non-dimensional form (omitting the asterisks for simplicity in writing):

$$\Lambda(D^2-a^2)^2 w - (p + D_a^{-1})(D^2-a^2)w$$
$$= Ra^2\theta - R_1 a^2\phi_1 - R_2 a^2\phi_2, \quad (36)$$

$$(D^2 - a^2 - E\sigma p)\theta = -w, \quad (37)$$

$$\left(D^2 - a^2 - \frac{E_1\sigma p}{\tau_1}\right)\phi_1 = -\frac{w}{\tau_1}, \quad (38)$$

$$\left(D^2 - a^2 - \frac{E_2\sigma p}{\tau_2}\right)\phi_2 = -\frac{w}{\tau_2}. \quad (39)$$

Eqs. (36)–(39) are to be solved by using the following boundary conditions:

$w = \theta = \phi_1 = \phi_2 = Dw = 0$ at $z = 0$ and at $z = 1$ (both the boundaries are rigid) (40)

or $w = \theta = \phi_1 = \phi_2 = D^2 w = 0$ at $z = 0$ and at $z = 1$ (both the boundaries are free) (41)

or $w = \theta = \phi_1 = \phi_2 = Dw = 0$ at $z = 0$, (lower boundary is rigid) and $w = \theta = \phi_1 = \phi_2 = D^2 w = 0$ at $z = 1$, (upper boundary is free) (42)

or $w = \theta = \phi_1 = \phi_2 = D^2 w = 0$ at $z = 0$, (lower boundary is free) and $w = \theta = \phi_1 = \phi_2 = Dw = 0$ at $z = 1$, (upper boundary is rigid), (43)

where z is the real independent such that $0 \leqslant z \leqslant 1$, D is the differentiation w.r.t. z, a^2 is square of the wave number, $\sigma > 0$ the Prandtl number, $\tau_1 > 0$ and $\tau_2 > 0$ are the Lewis numbers for two concentration components respectively, $D_a > 0$ is the Darcy number, $R > 0$ is the Rayleigh number, $R_1 > 0$ and $R_2 > 0$ are the two concentration Rayleigh numbers, $p = p_r + ip_i$ is the complex growth rate where p_r and p_i are the real constants, w is the vertical velocity, θ is the temperature, ϕ_1 and ϕ_2 are the respective concentrations of the two components, $E_1 > 0$ and $E_2 > 0$ are constants. It may further be noted that Eqs. (36)–(43) describe an eigenvalue problem for p and govern triply diffusive convection in a porous medium for any combination of dynamically free and rigid boundaries.

3. Mathematical analysis

Now we prove the following theorem:

Theorem. *If* $R > 0$, $R_1 > 0$, $R_2 > 0$, $p_r \geqslant 0, p_i \neq 0$, *then a necessary condition for the existence of nontrivial solution* $(w, \theta, \phi_1, \phi_2, p)$ *of Eqs.* (36)–(39) *together with either of the boundary conditions* (40)–(43) *is that*

$$|p|^2 < \frac{R_1}{E_1\sigma} + \frac{R_2}{E_2\sigma}.$$

Proof. Multiplying Eq. (36) by w^* (the superscript * henceforth denotes complex conjugation) on both sides and integrating over vertical range of z, we obtain

$$\Lambda\int_0^1 w^*(D^2-a^2)^2 w\,dz - (p + D_a^{-1})\int_0^1 w^*(D^2-a^2)w\,dz$$
$$= Ra^2\int_0^1 w^*\theta\,dz - R_1 a^2\int_0^1 w^*\phi_1\,dz - R_2 a^2\int_0^1 w^*\phi_2\,dz. \quad (44)$$

Making use of Eqs. (37)–(39) and the fact, that $w(0) = 0 = w(1)$, we can write

$$Ra^2\int_0^1 w^*\theta\,dz = -Ra^2\int_0^1 \theta(D^2 - a^2 - E\sigma p^*)\theta^*\,dz, \quad (45)$$

$$R_1 a^2\int_0^1 w^*\phi_1\,dz = -R_1 a^2\tau_1\int_0^1 \phi_1\left(D^2 - a^2 - \frac{E_1\sigma p^*}{\tau_1}\right)\phi_1^*\,dz, \quad (46)$$

$$R_2 a^2\int_0^1 w^*\phi_2\,dz = -R_2 a^2\tau_2\int_0^1 \phi_2\left(D^2 - a^2 - \frac{E_2\sigma p^*}{\tau_2}\right)\phi_2^*\,dz. \quad (47)$$

Combining Eqs. (44)–(47), we obtain

$$\Lambda\int_0^1 w^*(D^2-a^2)^2 w\,dz - (p + D_a^{-1})\int_0^1 w^*(D^2-a^2)w\,dz$$
$$= -Ra^2\int_0^1 \theta(D^2 - a^2 - E\sigma p^*)\theta^*\,dz$$
$$+ R_1 a^2\tau_1\int_0^1 \phi_1\left(D^2 - a^2 - \frac{E_1\sigma p^*}{\tau_1}\right)\phi_1^*\,dz$$
$$+ R_2 a^2\tau_2\int_0^1 \phi_2\left(D^2 - a^2 - \frac{E_2\sigma p^*}{\tau_2}\right)\phi_2^*\,dz. \quad (48)$$

Integrating various terms of Eq. (48) by parts for an appropriate number of times and making use of either of the boundary conditions (40)–(43), it follows that

$$\Lambda \int_0^1 \left(|D^2 w|^2 + 2a^2|Dw|^2 + a^4|w|^2\right) dz + (p + D_a^{-1}) \int_0^1 (|Dw|^2 + a^2|w|^2) dz = Ra^2 \int_0^1 (|D\theta|^2 + a^2|\theta|^2 + E\sigma p^*|\theta|^2) dz - R_1 a^2 \tau_1 \int_0^1 \left(|D\phi_1|^2 + a^2|\phi_1|^2 + \frac{E_1\sigma p^*}{\tau_1}|\phi_1|^2\right) dz - R_2 a^2 \tau_2 \int_0^1 \left(|D\phi_2|^2 + a^2|\phi_2|^2 + \frac{E_2\sigma p^*}{\tau_2}|\phi_2|^2\right) dz. \quad (49)$$

Equating imaginary parts of both sides of Eq. (49), and canceling $p_i(\neq 0)$ throughout, we have

$$\int_0^1 (|Dw|^2 + a^2|w|^2) dz = -Ra^2 E\sigma \int_0^1 |\theta|^2 dz + R_1 a^2 E_1 \sigma \int_0^1 |\phi_1|^2 dz + R_2 a^2 E_2 \sigma \int_0^1 |\phi_2|^2 dz. \quad (50)$$

Now, multiplying Eq. (38) by its complex conjugate and integrating the resulting equation for a suitable number of times and using the boundary condition on ϕ_1 namely, $\phi_1(0) = 0 = \phi_1(1)$, we obtain

$$\int_0^1 (|D^2\phi_1|^2 + 2a^2|D\phi_1|^2 + a^4|\phi_1|^2) dz + \frac{2E_1\sigma p_r}{\tau_1} \int_0^1 (|D\phi_1|^2 + a^2|\phi_1|^2) dz + \frac{E_1^2\sigma^2|p|^2}{\tau_1^2} \int_0^1 |\phi_1|^2 dz = \frac{1}{\tau_1^2} \int_0^1 |w|^2 dz. \quad (51)$$

Since $p_r \geqslant 0$, it follows from Eq. (51), that

$$\int_0^1 |\phi_1|^2 dz < \frac{1}{E_1^2 \sigma^2 |p|^2} \int_0^1 |w|^2 dz. \quad (52)$$

In the same manner, we obtain from Eq. (39), the inequality

$$\int_0^1 |\phi_2|^2 dz < \frac{1}{E_2^2 \sigma^2 |p|^2} \int_0^1 |w|^2 dz. \quad (53)$$

Utilizing inequalities (52) and (53) in Eq. (50), we have

$$\int_0^1 |Dw|^2 dz + a^2 \left(1 - \frac{R_1}{E_1\sigma|p|^2} - \frac{R_2}{E_2\sigma|p|^2}\right) \int_0^1 |w|^2 dz + Ra^2 E\sigma \int_0^1 |\theta|^2 dz < 0, \quad (54)$$

which clearly implies that

$$|p|^2 < \frac{R_1}{E_1\sigma} + \frac{R_2}{E_2\sigma} \quad (55)$$

This proves the theorem. □

The above theorem states, from the physical point of view, that 'the complex growth rate (p_r, p_i) of an arbitrary neutral or unstable oscillatory perturbation of growing amplitude, in a triply diffusive fluid layer saturating a porous medium with one of the components as heat with diffusivity κ, must lie inside a semicircle in the right- half of the (p_r, p_i)-plane whose center is the origin and radius equals $\sqrt{\frac{R_1}{E_1\sigma} + \frac{R_2}{E_2\sigma}}$, where R_1 and R_2 are the Rayleigh numbers for the two concentration components with diffusivities κ_1 and κ_2 (with no loss of generality, $\kappa > \kappa_1 > \kappa_2$), σ is the Prandtl number, E_1 and E_2 are constants. It is further proved that this result is uniformly valid for any combination of rigid and/or free boundaries.

4. Special case

The following result may be obtained from the above theorem as a special case.

For thermohaline convection in porous medium $(R_1 > 0, R_2 = 0)$, $|p| < \sqrt{\frac{R_1}{E_1\sigma}}$.

5. Conclusions

Linear stability theory is used to investigate triple diffusive convection in porous medium. The Darcy-Brinkman model has been used which is more compatible for the flow through high porosity medium. Upper bounds for the linear growth rate of an arbitrary neutral or unstable oscillatory perturbation of growing amplitude are obtained. Further, the result for double diffusive convection in porous medium is also obtained as a consequence.

Conflict of interest

There is no conflict of interest.

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Nomenclature

d	depth of the fluid layer
D_a	Darcy number
$E = \epsilon + (1-\epsilon)\frac{\rho_s C_s}{\rho_0 C_f}$	heat capacity ratio
E_1	a constant analogous to E corresponding to first concentration component
E_2	a constant analogous to E corresponding to second concentration component
σ	thermal Prandtl number
R	Rayleigh number
R_1	thermohaline Rayleigh number for first concentration component
R_2	thermohaline Rayleigh number for second concentration component
S_{10}	concentration of the first component at the lower boundary
S_{11}	concentration of the first component at the upper boundary
S_{20}	concentration of the second component at the lower boundary
S_{21}	concentration of the second component at the upper boundary
T_0	temperature at the lower boundary
T_1	temperature at the upper boundary
k_1	permeability of the porous medium
P	pressure
u, v, w	velocity components along x, y, z-directions respectively

Greek symbols

α	coefficient of thermal expansion
β	uniform temperature gradient
α_1	coefficient of mass expansion for first concentration component
α_2	coefficient of mass expansion for second concentration component
β_1	uniform concentration gradient for first concentration component
β_2	uniform concentration gradient for second concentration component
ϵ	porosity of the porous medium
ρ	density
μ_e	effective viscosity
μ	dynamic viscosity
τ	Lewis number
τ_1	Lewis number for first concentration component
τ_2	Lewis number for second concentration component
ν	kinematic viscosity
κ	thermal diffusivity
κ_1	mass diffusivity for first concentration component
κ_2	mass diffusivity for second concentration component
$\Lambda = \frac{\mu_e}{\mu}$	viscosity ratio

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