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ORIGINAL ARTICLE

A fractional model of fluid flow through porous media with mean capillary pressure

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Abstract In this paper, we discuss a fractional model arising in flow of two incompatible liquids through homogenous porous media with mean capillary pressure. The solution is derived by the application of the Sumudu transform and the Fourier sine transform. The results are received in compact and graceful forms in terms of the generalized Mittag-Leffler function, which are suitable for numerical computation. The mathematical formulation leads to generalized fractional derivative which has been solved by using a numerical technique by employing the iterative process with the help of appropriate boundary conditions. This problem has great importance in petroleum technology.

1. Introduction

A porous medium is a material containing pores (voids). Voids are usually filled with a fluid as liquid gas. A porous medium is most often characterized by its porosity. The skeletal portion of the material is often called the matrix or frame. Other properties of the medium such as permeability, electrical conductivity and tensile strength can also be consequent for the respective properties of its constituents (solid matrix and fluid) and the media porosity and pore structure, but these are generally complex. For a poroelastic medium the concept of porosity is usually uncomplicated. This concept of porous media is used in many areas of applied science and engineering. The oil–water movement in a porous medium is an important problem of petroleum technology and water hydrology (Scheidegger, 1966). Here we consider the injection of water into an oil formation in porous medium providing a two phase liquid–liquid flow problem. Such a problem is generally encountered in secondary recovery process. A number of research workers have also studied phenomenon of flow of two incompatible liquids through homogenous porous media with mean capillary pressure by using different mathematical resources (Bravo and Araujo, 2008; Brooks and Corey, 1964; Corey, 1954; Scheidegger, 1960; Scheidegger and Johnson, 1961). The fractional calculus has gained importance and popularity during the recent years or so, mainly due to its demonstrated applications in science and engineering. For example, these equations are increasingly used to model problems in fluid flow, theology, diffusion, relaxation, oscillation, anomalous diffusion, reaction–diffusion, turbulence, diffusive transport akin to

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diffusion, electric networks, polymer physics, chemical physics, electrochemistry of corrosion, relaxation processes in complex systems, propagation of seismic waves, dynamical processes in self-similar and porous structures and many other physical processes (Hilfer, 2000; Srivastava et al., 2012; Moustafa and Salem, 2006; Podlubny, 1999; He, 1998; Chaurasia and Singh, 2010). Many authors have proposed various methods to handle linear and non-linear fractional differential equations which are of great importance in scientific and technological fields. Among these are differential transform method (He, 1998; Atangana and Alabaraoye, 2013; Atangana and Kilicman, 2013), homotopy perturbation method (Liu et al., 2014), and variational iteration method (He and Wu, 2007).

In this article, we study a fractional partial differential equation associated with the generalized fractional derivative which is governed by the flow of immiscible phases in a homogenous porous medium with initial and boundary conditions. The solution of the fractional model is obtained by using Sumudu and Sine transforms.

2. Preliminary results

The Sumudu transform of a function $f(t)$, determined for all real numbers $t \geqslant 0$, is the function $F_s(u)$, defined by Watugala (1993), Weerakoon (1994), Asiru (2001), and Belgacem and Karaballi (2005).

$$S\{f(t)\} = \bar{F}(u) = G(u) = \int_0^\infty (1/u)e^{-t/u} f(t)dt. \tag{1}$$

We will also use the following outcome hold by Chaurasia and Singh (2011) as:

$$S^{-1}[u^{\gamma-1}(1-\omega u^\beta)^{-\delta}] = t^{\gamma-1}E^{\delta}_{\beta,\gamma}(\omega t^\beta). \tag{2}$$

The Fourier sine transform is defined by Debnath (1995).

$$\bar{F}(s,t) = \frac{2}{\sqrt{\pi}}\int_0^\infty f(x,t)\sin sx\, dx. \tag{3}$$

The error function of x is defined by Rainville (1960)

$$erf(x) = \frac{2}{\pi}\int_0^x \exp(-z^2)\, dz \tag{4}$$

and the complimentary error function of x is defined as:

$$erf_c(x) = \frac{2}{\pi}\int_0^x \exp(-z^2)\, dz \tag{5}$$

A generalization of the Mittag-Leffler function by Mittag-Leffler (1903, 1905)

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha+1)}, \quad (\alpha \in C, \mathbb{R}(\alpha) > 0) \tag{6}$$

was introduced (Wiman, 1905) in the general form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha+\beta)}, \quad (\alpha,\beta \in C, \mathbb{R}(\alpha) > 0) \tag{7}$$

also derived (Shukla and Prajapati, 2007) in the following integral:

$$\int_0^\infty e^{-st}t^{\beta-1}\frac{d^k}{dz^k}E_{\alpha,\beta}(xt^\alpha)dt = \frac{k!s^{\alpha-\beta}}{(s^\alpha - x)^{k+1}}. \tag{8}$$

The fractional derivative of order $\alpha > 0$ is presented (Caputo, 1967) in the form:

$$\begin{aligned} {}_0^cD_x^\alpha f(x) &= \frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha-m+1}}d\tau, \quad m-1<\alpha<m \\ &= \frac{d^m f(x)}{dx^m}, \quad \text{if } \alpha = m; m \in N \end{aligned} \tag{9}$$

where $\frac{d^m f(x)}{dx^m}$ is the m^{th} derivative of order m of the function $f(x)$ with respect to x. The Sumudu transform of this derivative is given (Chaurasia and Singh, 2010) as:

$$S[{}_0^cD_x^\alpha f(x); s] = u^{-\alpha}\bar{f}(s) - \sum_{k=0}^{m-1} u^{-\alpha+k}f^{(k)}(0+), \quad m-1<\alpha\leqslant m. \tag{10}$$

A generalization of the Caputo fractional derivative operator Eq. (9) is given (Hilfer, 2000), by introducing a right-sided fractional derivative operator of two parameters of order $0<\alpha<1$ and $0\leqslant\beta\leqslant 1$ as:

$${}_0D_{a+}^{\alpha,\beta}f(x) = I_{a+}^{\beta(1-\alpha)}\frac{d}{dx}\left(I_{a+}^{(1-\beta)(1-\alpha)}f(x)\right). \tag{11}$$

If we put $\beta = 1$, Eq. (11) reduces the Caputo fractional derivative operator assigned from Eq. (9).

Sumudu transform formula for this operator is given by Hilfer (2000), Belgacem et al. (2003), we hold:

$$S[{}_0D_x^{\alpha,\beta}f(x); s] = u^{-\alpha}\bar{f}(s) - u^{-\beta(\alpha-1)+1}I_{0+}^{(1-\beta)(1-\alpha)}f(0+); \quad 0<\alpha\leqslant 1, \tag{12}$$

where the initial value term

$$I_{0+}^{(1-\beta)(1-\alpha)}f(0+), \tag{13}$$

involves the Riemann–Liouville fractional integral operator of order $(1-\beta)(1-\alpha)$ evaluated in the limit as $x\to 0+$. For more details and properties of this operator see Tomovski et al. (2010).

The simplest Wright function is defined (Erdélyi et al., 1981; Srivastava et al., 2012) as:

$$\phi(\alpha,\beta;z) = \sum_{k=0}^\infty \frac{1}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}, \quad \text{where } \alpha,\beta,z \in C, \tag{14}$$

and the general Wright function is defined as:

$${}_p\psi_q(z) = {}_p\psi_q\left[\begin{matrix}(a_i,\alpha_i)_{(1,p)} \\ (b_j,\beta_j)_{(1,q)}\end{matrix}\middle| z\right] = \sum_{k=0}^\infty \frac{\prod_{i=1}^p \Gamma(a_i+\alpha_i k)}{\prod_{j=1}^q \Gamma(b_j+\beta_j k)}\frac{z^k}{k!}, \tag{15}$$

where $z, a_i, b_j \in C$ and $\alpha_i, \beta_j \in R (i = 1,2,\ldots,p$ and $j = 1,2,\ldots,q)$ then Eq. (15) reduces to familiar generalized hyper-geometric function as (Thomas and George, 2006)

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = \sum_{k=0}^\infty \frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{z^k}{k!}. \tag{16}$$

The generalized Navier–Stokes equations are given as (Moustafa and Salem, 2006)

$$\mathcal{W}(\alpha,\beta;z) = \sum_{k=0}^\infty \frac{1}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}, \quad \text{where } \alpha,\beta,z \in C. \tag{17}$$

The relationship between the Wright function and the complementary Error function is given as

$$\mathcal{W}\left(-\frac{1}{2},1;z\right)=erf_c\left(\frac{z}{2}\right). \tag{18}$$

3. Mathematical model of the problem

The seepage velocity (U_w) of water and oil (ζ_w) is assumed as (Scheidegger, 1960)

$$U_w=-\frac{K_w}{\zeta_w}K\frac{\partial P_w}{\partial x} \tag{19}$$

$$U_o=-\frac{K_o}{\zeta_o}K\frac{\partial P_o}{\partial x} \tag{20}$$

and equation of continuity

$$\psi\frac{\partial S_w}{\partial t}+\frac{\partial U_w}{\partial x}=0 \tag{21}$$

$$\psi\frac{\partial S_o}{\partial t}+\frac{\partial U_o}{\partial x}=0 \tag{22}$$

here K is considered as the permeability of the consistent medium, K_w and K_o are the relative permeability of water and oil, which are the functions of the saturation of water (S_w) and oil (S_o) respectively, P_w and P_o define the pressure of water and oil, aspect ζ_w and ζ_o are the kinematics viscosities of water and oil, while ψ is the medium of porosity and from the definition of phase saturation (Scheidegger, 1960), it is apparent that:

$$S_w+S_o=1 \tag{23}$$

The capillary pressure (P_c) is defined as the pressure discontinuity of the flowing phases across their common interface which may be codified as:

$$P_c=P_o-P_w. \tag{24}$$

Relation between phase saturation and relative permeability (Scheidegger and Johnson, 1961) is specified as:

$$\left.\begin{aligned}K_w&=S_w\\K_o&=1-S_w\\K_o&=S_o\end{aligned}\right\} \tag{25}$$

If the generalized fractional derivative model is used to present the time derivative term, the equation of continuity is transformed into:

$$\psi D_t^{\alpha,\beta}S_w+\frac{\partial U_w}{\partial x}=0;\quad \text{where } (0<\alpha\leqslant 1;0<\beta\leqslant 1), \tag{26}$$

$$\psi D_t^{\alpha,\beta}S_o+\frac{\partial U_o}{\partial x}=0;\quad \text{where } (0<\alpha\leqslant 1;0<\beta\leqslant 1). \tag{27}$$

If we put $\alpha=1$ and $\beta=1$ in Eqs. (26) and (27) reduced to Eqs. (21) and (22) respectively.

4. Formulation of fractional partial differential equation

Putting the values of U_w and U_o in Eqs. (26) and (27) from Eqs. (19) and (20) respectively, we obtain the results:

$$\psi D_t^{\alpha,\beta}S_w=\frac{\partial}{\partial x}\left\{\frac{K_w}{\zeta_w}K\frac{\partial P_w}{\partial x}\right\};\quad \text{where } (0<\alpha\leqslant 1;0<\beta\leqslant 1), \tag{28}$$

$$\psi D_t^{\alpha,\beta}S_o=\frac{\partial}{\partial x}\left\{\frac{K_o}{\zeta_o}K\frac{\partial P_o}{\partial x}\right\};\quad \text{where } (0<\alpha\leqslant 1;0<\beta\leqslant 1). \tag{29}$$

Eliminating $\frac{\partial P_w}{\partial x}$ from Eq. (28), we get:

$$\psi D_t^{\alpha,\beta}S_w=\frac{\partial}{\partial x}\left\{\frac{K_w}{\zeta_w}K\left(\frac{\partial P_o}{\partial x}-\frac{\partial P_c}{\partial x}\right)\right\};$$

where $(0<\alpha\leqslant 1;0<\beta\leqslant 1)$. (30)

From Eqs. (29), (30) and (23) we get,

$$\frac{\partial}{\partial x}\left[K\left(\frac{K_w}{\zeta_w}+\frac{K_o}{\zeta_o}\right)\frac{\partial P_o}{\partial x}-K\frac{K_w}{\zeta_w}\frac{\partial P_c}{\partial x}\right]=0. \tag{31}$$

Now integrating Eq. (31), we get:

$$K\left(\frac{K_w}{\zeta_w}+\frac{K_o}{\zeta_o}\right)\frac{\partial P_o}{\partial x}-K\frac{K_w}{\zeta_w}\frac{\partial P_c}{\partial x}=-C, \tag{32}$$

where C is the constant of integration, whose value can be calculated.

$$\frac{\partial P_o}{\partial x}=\frac{-C}{K\left(\frac{K_w}{\zeta_w}+\frac{K_o}{\zeta_o}\right)}+\frac{K\frac{K_w}{\zeta_w}\frac{\partial P_c}{\partial x}}{K\left(\frac{K_w}{\zeta_w}+\frac{K_o}{\zeta_o}\right)}$$

$$\frac{\partial P_o}{\partial x}=\frac{-C}{K\frac{K_w}{\zeta_w}\left(1+\frac{K_o}{\zeta_o}\frac{\zeta_w}{K_w}\right)}+\frac{\frac{\partial P_c}{\partial x}}{\left(1+\frac{K_o}{\zeta_o}\frac{\zeta_w}{K_w}\right)}, \tag{33}$$

Substituting the value of $\frac{\partial P_o}{\partial x}$ in Eq. (30) from Eq. (33), we got the conclusion that:

$$\psi D_t^{\alpha,\beta}S_w=\frac{\partial}{\partial x}\left\{\frac{K_w}{\zeta_w}K\left(\frac{-C}{K\frac{K_w}{\zeta_w}\left(1+\frac{K_o}{\zeta_o}\frac{\zeta_w}{K_w}\right)}+\frac{\frac{\partial P_c}{\partial x}}{\left(1+\frac{K_o}{\zeta_o}\frac{\zeta_w}{K_w}\right)}\right)-\frac{K_w}{\zeta_w}K\frac{\partial P_c}{\partial x}\right\}$$

$$\psi D_t^{\alpha,\beta}S_w+\frac{\partial}{\partial x}\left[\frac{\frac{K_o}{\zeta_o}K\frac{\partial P_c}{\partial x}}{\left(1+\frac{K_o}{\zeta_o}\frac{\zeta_w}{K_w}\right)}+\frac{C}{\left(1+\frac{K_o}{\zeta_o}\frac{\zeta_w}{K_w}\right)}\right]=0 \tag{34}$$

Pressure of oil P_o can be defined as:

$$P_o=\frac{P_o+P_w}{2}+\frac{P_o-P_w}{2}=\bar{P}+\frac{1}{2}P_c. \tag{35}$$

where $\bar{P}$ is constant, which is the mean pressure.

From Eqs. (32) and (35), we hold:

$$C=\frac{K}{2}\left(\frac{K_w}{\zeta_w}-\frac{K_o}{\zeta_o}\right)\frac{\partial P_c}{\partial x}. \tag{36}$$

Substituting the value of C in Eq. (34), we hold:

$$\psi D_t^{\alpha,\beta}S_w+\frac{\partial}{\partial x}\left[\frac{\frac{K_o}{\zeta_o}K\frac{\partial P_c}{\partial x}}{\left(1+\frac{K_o}{\zeta_o}\frac{\zeta_w}{K_w}\right)}+\frac{\frac{K}{2}\left(\frac{K_w}{\zeta_w}-\frac{K_o}{\zeta_o}\right)\frac{\partial P_c}{\partial x}}{\left(1+\frac{K_o}{\zeta_o}\frac{\zeta_w}{K_w}\right)}\right]=0$$

$$\psi D_t^{\alpha,\beta}S_w+\frac{1}{2}\frac{\partial}{\partial x}\left[\frac{\frac{K_o}{\zeta_o}K\frac{\partial P_c}{\partial x}+\frac{K_w}{\zeta_w}K\frac{\partial P_c}{\partial x}}{\left(1+\frac{K_o}{\zeta_o}\frac{\zeta_w}{K_w}\right)}\right]=0$$

$$\psi D_t^{\alpha,\beta}S_w+\frac{1}{2}\frac{\partial}{\partial x}\left[K\frac{K_w}{\zeta_w}\frac{\partial P_c}{\partial x}\right]=0$$

$$\psi D_t^{\alpha,\beta}S_w+\frac{1}{2}\frac{\partial}{\partial x}\left[K\frac{K_w}{\zeta_w}\frac{\partial P_c}{\partial S_w}\frac{\partial S_w}{\partial x}\right]=0. \tag{37}$$

Taking the $K\frac{K_w}{\zeta_w}\frac{\partial P_c}{\partial S_w} = -B$ then Eq. (37) is reduced as:

$$\psi D_t^{\alpha,\beta} S_w - \frac{1}{2} B \frac{\partial^2 S_w}{\partial x^2} = 0$$

$$\frac{\partial^2 S_w}{\partial x^2} = \frac{1}{\mu} D_t^{\alpha,\beta} S_w, \tag{38}$$

where $C = \frac{B}{2P}$.
This is the partial differential equation of motion for saturation constraint conditions as:

$$\left.\begin{array}{l} S_w(x,0) = 0 \\ S_w(0,T) = S_w < 1 \\ \lim_{x\to\infty} S_w(x,T) = 0; 0 < x < \infty \end{array}\right\} \tag{39}$$

5. Analytical solution of the problem

Form Eq. (38), we include:

$$D_t^{\alpha,\beta} S_w(x,t) = \mu \frac{\partial^2 S_w}{\partial x^2}. \tag{40}$$

Operating Fourier sine transform Eq. (3) on Eq. (40), we get:

$$D_t^{\alpha,\beta} \bar{S}_w(s,t) = \mu\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 S_w}{\partial x^2} \sin sx\, dx$$

Integrating by parts method, yields

$$\begin{aligned} D_t^{\alpha,\beta} \bar{S}_w(s,t) &= \mu\sqrt{\frac{2}{\pi}}\left[\frac{\partial S_w}{\partial x}\sin sx\right]_0^\infty - s\mu\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial S_w}{\partial x}\cos sx\, dx \\ &= \mu\sqrt{\frac{2}{\pi}}\left[\frac{\partial S_w}{\partial x}\sin sx\right]_0^\infty - s\mu\sqrt{\frac{2}{\pi}}[S_w \cos sx]_0^\infty \\ &\quad - s^2\mu\sqrt{\frac{2}{\pi}}\int_0^\infty S_w \sin sx\, dx, \end{aligned}$$

and applying constraint conditions Eq. (40), we get:

$$\begin{aligned} D_t^{\alpha,\beta} \bar{S}_w(s,t) &= \mu\sqrt{\frac{2}{\pi}}(0) - s\mu\sqrt{\frac{2}{\pi}}(0 - S_{w_0}) - s^2\mu\bar{S}_w(s,t) \\ &= s\mu\sqrt{\frac{2}{\pi}} S_{w_0} - s^2\mu\bar{S}_w(s,t). \end{aligned} \tag{41}$$

Now, using Eq. (12) and taking Sumudu transform for both sides in Eq. (41), we obtain:

$$\tilde{\bar{S}}_w(s,u) = s\mu\sqrt{\frac{2}{\pi}} S_{w_0}[u^\alpha\{1 - (-s^2\mu)u^\alpha\}^{-1}]. \tag{42}$$

Next, using Eq. (2) and taking inverse Sumudu transform for both sides in Eq. (42), we obtain:

$$\bar{S}_w(s,t) = s\mu\sqrt{\frac{2}{\pi}} S_{w_0} t^\alpha E_{\alpha,\alpha+1}(-s^2\mu t^\alpha). \tag{43}$$

Finally, taking the inverse Fourier sine transform on above Eq. (43), we get:

$$S_w(x,t) = \mu\sqrt{\frac{2}{\pi}} S_{w_0} t^\alpha \left[\sqrt{\frac{2}{\pi}}\int_0^\infty \{sE_{\alpha,\alpha+1}(-s^2\mu t^\alpha)\}\sin sx\, ds\right],$$

$$S_w(x,t) = \frac{2}{\pi}\mu S_{w_0} t^\alpha \int_0^\infty \{sE_{\alpha,\alpha+1}(-s^2\mu t^\alpha)\}\sin sx\, ds \tag{44}$$

which is the same solution as recently obtained (Prajapati et al., 2012).

It can be written in terms of Wright function as:

$$S_w(x,t) = S_{w_0} W\left(\frac{-\alpha}{2}, 1; \frac{-x}{\sqrt{\mu t^\alpha}}\right). \tag{45}$$

If we set $\alpha = 1$ and make use of Eq. (12) and Eq. (45), we arrive at the following result:

$$S_w(x,t) = S_{w_0} erf_c\left(\frac{x}{2\sqrt{\mu t}}\right). \tag{46}$$

6. Conclusions

In this paper, we have presented a fractional model of flow of two incompatible liquids through homogenous porous media with mean capillary pressure. The solution has been developed in terms of Mittag-Leffler function by using the Sumudu transform and Fourier sine transform and their inverses after deriving the related formulae for fractional integrals and derivatives.

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References

Asiru, M.A., 2001. Sumudu transform and the solution of integral equation of convolution type. Int. J. Math. Educ. Sci. Technol. 32, 906–910.

Atangana, A., Alabaraoye, E., 2013. Solving system of fractional partial differential equations arisen in the model of HIV infection of CD4+ cells and attractor one-dimensional Keller-Segel equation. Adv. Differ. Equ. 2013 (article 94).

Atangana, A., Kilicman, A., 2013. The use of Sumudu transform for solving certain nonlinear fractional heat-like equations. Abstr. Appl. Anal. 2013, 12 (Article ID 737481).

Belgacem, F.B.M., Karaballi, A.A., 2005. Sumudu transform fundamental properties investigations and applications. Int. J. Appl. Math. Stoch. Anal., 1–23

Belgacem, F.B.M., Karaballi, A.A., Kalla, S.L., 2003. Analytical investigations of the Sumudu transform and applications to integral production equations. Math. Probl. Eng. 3, 103–118.

Bravo, M.C., Araujo, M., 2008. Analysis of the unconventional behavior of oil relative permeability during depletion tests of gas-saturated heavy oils. Int. J. Multiph. Flow 34 (5), 447–460.

Brooks, R.H., Corey, A.T., 1964. Hydraulic Properties of Porous Media. Hydrological Papers (Colorado State University) 3.

Caputo, M., 1967. Linear models of dissipation whose Q is almost frequency independent-II. Geophys. J. Royal Astron. Soc. 13, 529–539 (Reprinted. In: Fract. Calc. Appl. Anal., 11(1), 2008, 3–14).

Chaurasia, V.B.L., Singh, J., 2010. Application of Sumudu transform in schrodinger equation occurring in Quantum Mechanics. Appl. Math. Sci. 4 (57), 2843–2850.

Chaurasia, V.B.L., Singh, J., 2011. Application of Sumudu transform in fractional kinetic equations. Gen. Math. Notes 2 (1), 86–95.

Corey, A.T., 1954. The interrelation between gas and oil relative permeabilities. Prod. Monthly 19 (1), 38–41.

Debnath, L., 1995. Integral Transforms and Their Applications. CRC Press, New York, London, Tokyo.

Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F., 1981. Higher Transcendental Functions. Krieger, Melbourne, 3.

He, J.H., 1998. Approximate analytical solution for seepage flow with fractional derivatives in porous media. Comput. Meth. Appl. Mech. Eng. 167, 57–68.

He, J.H., Wu, X.H., 2007. Variational iteration method: new development and applications. Comput. Math. Appl. 54, 881–894.

Hilfer, R., 2000. Applications of Fractional Calculus in Physics. World Scientific Publishing Company, Singapore, New Jersey, Hong Kong (pp. 87–130).

Liu, H.Y., He, J.H., Li, Z.B., 2014. Fractional calculus for nanoscale flow and heat transfer. Int. J. Numer. Meth. Heat Fluid Flow 24 (6), 1227–1250.

Mittag-Leffler, G.M., 1903. Sur la nouvelle function $E\alpha\ (x)$. CR Acad. Sci. Paris 137, 554–558 (ser. II).

Mittag-Leffler, G.M., 1905. Sur la representatin analytique d'une branche uniforme d'une function monogene. Acta Math. 29, 101–181.

Moustafa, El.-Shahed, Salem, A., 2006. Decay of vortex velocity and diffusion of the temperature for fractional viscoelastic fluid through porous medium. Int. J. Commun. Heat Mass Transfer 33, 240–248.

Podlubny, I., 1999. Fractional Differential Equations. Academic Press, New York.

Prajapati, J.C., Patel, A.D., Pathak, K.N., Shukla, A.K., 2012. Fractional calculus approach in the study of instability phenomenon in fluid dynamics. Palestine J. Math. 1 (2), 95–103.

Rainville, E.D., 1960. Special Functions. The Macmillan Company, New York.

Scheidegger, A.E., 1960. The Physics Flow through Porous Media, 216. University of Toronto Press (pp. 229–231).

Scheidegger, A.E., 1966. Flow through Porous Media, as in Applied Mechanics Surveys by Abramson. Spartan Books, Washington.

Scheidegger, A.E., Johnson, E.F., 1961. The statistical behaviour of instabilities in displacement process in porous media. Can. J. Phys. 39 (2), 326.

Shukla, A.K., Prajapati, J.C., 2007. On a generalization of Mittag-Leffler function and its properties. J. Math. Anal. Appl. 336, 797–811.

Srivastava, H.M., Parmar, R.K., Chopra, P., 2012. A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions. Axioms 1 (3), 238–258.

Thomas, C., George, C., 2006. The Fox–Wright functions and Laguerre multiplier sequences. J. Math. Anal. Appl. 314, 109–125.

Tomovski, Z., Hilfer, R., Srivastava, H.M., 2010. Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions. Integral Transforms Spec. Funct. 21 (11), 797–814.

Watugala, G.K., 1993. Sumudu transform a new integral transform to solve differential equations and control engineering problems. Int. J. Math. Educ. Sci. Technol. 24, 35–43.

Weerakoon, S., 1994. Applications of Sumudu transform to partial differential equations. Int. J. Math. Educ. Sci. Technol. 25, 277–283.

Wiman, A., 1905. Uber den fundamentalsatz in der theorie der functionen $E\alpha\ (x)$. Acta Math. 29, 191–201.