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ORIGINAL ARTICLE

Optimal homotopy asymptotic method for solving *n*th order linear fuzzy initial value problems

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Abstract In this paper the optimal homotopy asymptotic method (OHAM) is employed to obtain approximate analytical solution of *n*th order ($n \geqslant 2$) linear fuzzy initial value problems (FIVPs). The convergence theorem of this method in fuzzy case is presented and proved. This method provides us with a convenient way to control the convergence of approximation series. The method is tested on *n*th linear FIVPs and comparisons of the exact solution that were made with numerical results showed the effectiveness and accuracy of this method.

1. Introduction

Many dynamical real life problems may be formulated as a mathematical model. These problems can be formulated either as a system of ordinary or partial differential equations. Fuzzy differential equations are a useful tool to model a dynamical system when information about its behavior is inadequate. Fuzzy ordinary differential equations may arise in the mathematical modeling of real world problems in which there is some uncertainty or vagueness. Fuzzy initial value problems (FIVPs) appear when the modeling of these problems was imperfect and its nature is under uncertainty. Fuzzy initial value problems arise in several areas of mathematics and science including population models (Ahmad and De Baets, 2009; Omer and Omer, 2013), mathematical physics (El Naschie, 2005) and medicine (Abbod et al., 2001; Barro and Marin, 2002) and other applications (Sahu and Saha Ray, 2015). Approximate-analytical methods such as the Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM) and Variational Iteration Method (VIM) have been used to solve fuzzy initial value problems involving ordinary differential equations. Ghanbari (2009) utilized HPM to solve first order linear fuzzy initial value problems. The ADM was employed by Babolian et al. (2004) and Allahviranlo et al. (2008) to solve first order linear and nonlinear fuzzy initial value problems. Abbasbandy et al. (2011) used the VIM to solve linear systems of first order fuzzy initial value problem.

OHAM is somewhat different from other approximate-analytical methods in that it gives extremely good results for even a large domain with minimal terms of the approximate series solution. In OHAM, the control and adjust of the convergence region are provided in a convenient way. Moreover, OHAM is also parameter free and provides better accuracy over the approximate analytical methods at the same order of approximation.

OHAM was introduced recently by Marinca et al. (2008) and applied for solving nonlinear problems without depending on the small parameter (Herisanu et al., 2008; Marinca and

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Herisanu, 2008; Marinca et al., 2009). Mabood (2014) has provided a comparative study between OHAM and HPM for strongly nonlinear equation. In this study, it was observed that OHAM gives an accurate solution as compared to HPM. Moreover, an advantage of OHAM is that it does not need any initial guess or to identify the $\hbar$-curve like Homotopy Analysis Method (HAM) and it is also parameter free. Furthermore, the OHAM has built in convergence criteria similar to HAM but with a greater degree of flexibility (Iqbal et al., 2010). The proposed method (OHAM) has also been successfully applied on various engineering problems (Alomari et al., 2013a,b; Anakira et al., 2013; Mabood et al., 2013a,b; 2014a,b; Herisanu et al., 2015; Marinca and Herisanu, 2014).

In this paper, our aim is to apply OHAM to nth order ($n \geqslant 2$) FIVP directly without reducing it to a system. To the best of our knowledge, this is the first attempt at solving a high order FIVP using the OHAM with proof of the convergence in fuzzy case. The outline of this paper is as follows. We will start in Section 2 with some preliminary concepts about fuzzy numbers. In Section 3 we reviewed the concept of OHAM and formulated it to obtain a reliable approximate solution to n$^{\text{th}}$ order FIVPs. In Section 4, the convergence theorem of OHAM is presented and proved, In Section 5, we consider numerical examples to show the capability of this method and finally, in Section 6, we give the conclusions of this study.

2. Preliminaries

Definition 2.1 (Bodjanova, 2006). The r-level (or r-cut) set of a fuzzy set $\tilde{A}$, labeled as $\tilde{A}_r$, is the crisp set of all $x \in X$ such that $\mu_{\tilde{A}} \geqslant r$ i.e.

$$\tilde{A}_r = \{x \in X | \mu_{\tilde{A}} > r, \quad r \in [0,1]\}$$

Definition 2.2. Fuzzy numbers are a subset of the real numbers set, and represent uncertain values. Fuzzy numbers are linked to degrees of membership which state how true it is to say if something belongs or not to a determined set. A fuzzy number (Dubois and Prade, 1982) μ is called a triangular fuzzy number if defined by three numbers $\alpha < \beta < \gamma$ where the graph of $\mu(x)$ is a triangle with the base on the interval $[\alpha, \beta]$ and vertex at $x = \beta$, and its membership function has the following form: (see Fig. 1)

$$\mu(x;\alpha,\beta,\gamma) = \begin{cases} 0, & \text{if} \quad x < \alpha \\ \frac{x-\alpha}{\beta-\alpha}, & \text{if} \quad \alpha \leqslant x \leqslant \beta \\ \frac{\gamma-x}{\gamma-\beta}, & \text{if} \quad \beta \leqslant x \leqslant y \\ 0, & \text{if} \quad x > \gamma \end{cases}$$

and its r-level is $[\mu]_r = [\alpha + r(\beta - \alpha), \gamma - r(\gamma - \beta)], r \in [0,1]$.

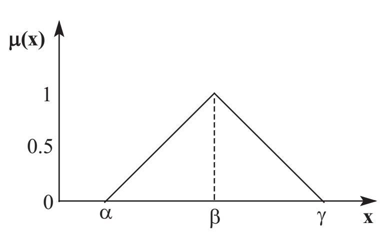

Figure 1 Triangular fuzzy number.

In this paper the class of all fuzzy subsets of $\mathbb{R}$ will be denoted by $\tilde{E}$ and satisfy the following properties (Dubois and Prade, 1982; Mansouri and Ahmady, 2012):

1. $\mu(t)$ is normal, i.e. $\exists\, t_0 \in \mathbb{R}$ with $\mu(t_0) = 1$.
2. $\mu(t)$is convex fuzzy set, i.e. $\mu(\lambda t + (1-\lambda)s) \geqslant min\{\mu(t), \mu(s)\}\ \forall t, s \in \mathbb{R}, \lambda \in [0,1]$.
3. μ upper semi-continuous on $\mathbb{R}$, and $\overline{\{t \in \mathbb{R} : \mu(t) > 0\}}$ is compact.

$\tilde{E}$ is called the space of fuzzy numbers and $\mathbb{R}$ is a proper subset of $\tilde{E}$.

Define the r-level set $x \in \mathbb{R}$, $[\mu]_r = \{x \setminus \mu(x) \geqslant r\}$, $0 \leqslant r \leqslant 1$ where $[\mu]_0 = \{x \setminus \mu(x) > 0\}$ is compact which is a closed bounded interval and denoted by $[\mu]_r = (\underline{\mu}(t), \bar{\mu}(t))$. In the parametric form, a fuzzy number is represented by an ordered pair of functions $(\underline{\mu}(t), \bar{\mu}(t))$, which satisfies (Kaleva, 1987):

1. $\underline{\mu}(t)$ is a bounded left continuous non-decreasing function over $[0,1]$.
2. $\bar{\mu}(t)$ is a bounded left continuous non-increasing function over $[0,1]$.
3. $\underline{\mu}(t) \leqslant \bar{\mu}(t)$, $r \in [0,1]$.

A crisp number r is simply represented by $\underline{\mu}_{(r)} = \bar{\mu}(r) = r$, $r \in [0,1]$.

Definition 2.3 (Seikkala, 1987). If $\tilde{E}$ be the set of all fuzzy numbers, we say that $f(t)$ is a fuzzy function if $f : \mathbb{R} \to \tilde{E}$.

Definition 2.4 (Fard, 2009). A mapping $f : T \to \tilde{E}$ for some interval $T \subseteq \tilde{E}$ is called a fuzzy function process and we denote r-level set by:

$$[\tilde{f}(t)]_r = [\underline{f}(t;r), \bar{f}(t;r)], \quad t \in T, \ r \in [0,1]$$

The r-level sets of a fuzzy number are much more effective as representation forms of fuzzy set than the above. Fuzzy sets can be defined by the families of their r-level sets based on the resolution identity theorem.

Definition 2.5 (Kaleva, 1987). The fuzzy integral of fuzzy process, $\tilde{f}(t;r)$, $\int_a^b \tilde{f}(t;r)dt$ for $a, b \in T$ and $r \in [0,1]$ is defined by:

$$\int_a^b \tilde{f}(t;r)dt = \left[\int_a^b \underline{f}(t;r)dt, \int_a^b \bar{f}(t;r)dt\right]$$

Definition 2.6 (Zadeh, 1965). Each function $f : X \to Y$ induces another function $\tilde{f} : F(X) \to F(Y)$ defined for each fuzzy interval U in X by:

$$\tilde{f}(U)(y) = \begin{cases} Sup_{x \in f^{-1}(y)} U(x), & \text{if} \quad y \in range(f) \\ 0, & \text{if} \quad y \notin range(f) \end{cases}$$

This is called the Zadeh extension principle.

Definition 2.7 (Salahshour, 2011). Consider $\tilde{x}, \tilde{y} \in \tilde{E}$. If there exists $\tilde{z} \in \tilde{E}$ such that $\tilde{x} = \tilde{y} + \tilde{z}$, then $\tilde{z}$ is called the H-difference (Hukuhara difference) of x and y and is denoted by $\tilde{z} = \tilde{x} \ominus \tilde{y}$.

Definition 2.8 (Zadeh, 2005). If $\tilde{f} : I \to \tilde{E}$ and $y_0 \in I$, where $I \in [t_0, T]$. We say that $\tilde{f}$ Hukuhara Differentiable at y_0, if there exists an element $[\tilde{f}']_r \in \tilde{E}$ such that for all $h > 0$ sufficiently small (near to 0), exists $\tilde{f}(y_0 + h; r) \ominus \tilde{f}(y_0; r)$, $\tilde{f}(y_0; r) \ominus \tilde{f}(y_0 - h; r)$ and the limits are taken in the metric $(\tilde{E}, D)$

$$\lim_{h \to 0+} \frac{\tilde{f}(y_0 + h; r) \ominus \tilde{f}(y_0; r)}{h} = \lim_{h \to 0+} \frac{\tilde{f}(y_0; r) \ominus \tilde{f}(y_0 - h; r)}{h}$$

The fuzzy set $[\tilde{f}'(y_0)]_r$ is called the Hukuhara derivative of $[\tilde{f}']_r$ at y_0.

These limits are taken in the space $(\tilde{E}, D)$ if t_0 or T, then we consider the corresponding one-side derivation. Recall that $\tilde{x} \ominus \tilde{y} = \tilde{z} \in \tilde{E}$ are defined on r-level set, where $[\tilde{x}]_r \ominus [\tilde{y}]_r = [\tilde{z}]_r$, $\forall r \in [0, 1]$. By consideration of definition of the metric D all the r-level sets $[\tilde{f}(0)]_r$ are Hukuhara differentiable at y_0, with Hukuhara derivatives $[\tilde{f}'(y_0)]_r$, when $\tilde{f} : I \to \tilde{E}$ is Hukuhara differentiable at y_0 with Hukuhara derivative $[\tilde{f}'(y_0)]_r$ and it leads to that $\tilde{f}$ is Hukuhara differentiable for all $r \in [0, 1]$ which satisfies the above limits i.e. if f is differentiable at $t_0 \in [t_0 + \alpha, T]$ then all its r-levels $[\tilde{f}'(t)]_r$ are Hukuhara differentiable at t_0.

Definition 2.9 (Salahshour, 2011). Define the mapping $\tilde{f}' : I \to \tilde{E}$ and $y_0 \in I$, where $I \in [t_0, T]$. We say that $\tilde{f}'$ is Hukuhara differentiable $t \in \tilde{E}$, if there exists an element $[\tilde{f}^{(n)}]_r \in \tilde{E}$ such that for all $h > 0$ sufficiently small (near to 0), exist $\tilde{f}^{(n-1)}(y_0 + h; r) \ominus \tilde{f}^{(n-1)}(y_0; r)$, and $\tilde{f}^{(n-1)}(y_0; r) \ominus \tilde{f}^{(n-1)}(y_0 - h; r)$ and the limits are taken in the metric $(\tilde{E}, D)$

$$\lim_{h \to 0+} \frac{\tilde{f}^{(n-1)}(y_0 + h; r) \ominus \tilde{f}^{(n-1)}(y_0; r)}{h} = \lim_{h \to 0+} \frac{\tilde{f}^{(n-1)}(y_0; r) \ominus \tilde{f}^{(n-1)}(y_0 - h; r)}{h}$$

exists and equal to $\tilde{f}^{(n)}$ and for $n = 2$ we have second order Hukuhara derivative.

Theorem 2.1 Mansouri and Ahmady, 2012. Let $\tilde{f} : [t_0 + \alpha, T] \to \tilde{E}$ be Hukuhara differentiable denoted by

$$[\tilde{f}'(t)]_r = \left[\underline{f}'(t), \bar{f}'(t)\right]_r = \left[\underline{f}'(t; r), \bar{f}'(t; r)\right]$$

Then the boundary functions $\underline{f}'(t; r), \bar{f}'(t; r)$ are differentiable

$$[\tilde{f}'(t)]_r = \left[(\underline{f}(t; r))', (\bar{f}(t; r))'\right], \quad \forall r \in [0, 1]$$

Theorem 2.2 Xiaobin and Dequan, 2013. Let $\tilde{f} : [t_0 + \alpha, T] \to \tilde{E}$ be Hukuhara differentiable denote by

$$[\tilde{f}'(t)]_r = \left[\underline{f}'(t), \bar{f}'(t)\right]_r = \left[\underline{f}'(t; r), \bar{f}'(t; r)\right]$$

When the boundary functions $\underline{f}'(t; r), \bar{f}'(t; r)$ are differentiable we can write for nth order fuzzy derivative

$$[\tilde{f}^{(n)}(t)]_r = \left[(\underline{f}^{(n)}(t; r))', (\bar{f}^{(n)}(t; r))'\right], \quad \forall r \in [0, 1]$$

3. Fuzzification and defuzzification of OHAM

The general structure of OHAM for solving crisp nth order ordinary differential equations was described in Gupta and Saha Ray (2014), Herisanu et al. (2015), Marinca and Herisanu (2014). To solve the nth order FIVP, there is a need to fuzzify and then defuzzify OHAM. Consider the following general nth order FIVP

$$\tilde{y}^{(n)}(t) = \tilde{f}\left(t, \tilde{y}(t), \tilde{y}'(t), \tilde{y}''(t), \ldots \tilde{y}^{(n-1)}(t)\right) + \tilde{w}(t), \quad t \in [t_0, T] \tag{1}$$

subject to the initial conditions

$$\tilde{y}(t_0) = \tilde{y}_0, y'(t_0) = \tilde{y}'_0, \ldots y^{(n-1)}(t_0) = \tilde{y}_0^{(n-1)} \tag{2}$$

where $\tilde{y}$ is a fuzzy function of the crisp variable t with $\tilde{f}$ being a fuzzy function of the crisp variable t, the fuzzy variable $\tilde{y}$ and the fuzzy Hukuhara-derivatives $\tilde{y}'(t), \tilde{y}''(t), \ldots \tilde{y}^{(n-1)}(t)$. Here $y^{(n)}$ is the fuzzy nth order Hukuhara-derivative and $\tilde{y}(t_0), \tilde{y}'(t_0), \ldots \tilde{y}^{(n-1)}(t_0)$ are convex fuzzy numbers. We denote the fuzzy function y by $\tilde{y} = [\underline{y}, \bar{y}]$ for $t \in [t_0, T]$ and $r \in [0, 1]$. It means that the r-level set of $\tilde{y}(t)$ can be defined as:

$$[\tilde{y}(t)]_r = \left[\underline{y}(t; r), \bar{y}(t; r)\right]$$

$$[\tilde{y}'(t)]_r = \left[\underline{y}'(t; r), \bar{y}'(t_0; r)\right], \ldots [\tilde{y}^{(n-1)}(t)]_r = \left[\underline{y}^{(n-1)}(t; r), \bar{y}^{(n-1)}(t; r)\right]$$

$$[\tilde{y}(t_0)]_r = \left[\underline{y}(t_0; r), \bar{y}(t_0; r)\right],$$

$$[\tilde{y}'(t_0)]_r = \left[\underline{y}'(t_0; r), \bar{y}'(t_0; r)\right], \ldots, [\tilde{y}^{(n-1)}(t_0)]_r = \left[\underline{y}^{(n-1)}(t_0; r), \bar{y}^{(n-1)}(t_0; r)\right]$$

where the fuzzy inhomogeneous term is $[\tilde{w}(t)]_r = [\underline{w}(t; r), \bar{w}(t; r)]$.

Since $\tilde{y}^{(n)}(t) = \tilde{f}\left(t, \tilde{y}(t), \tilde{y}'(t), \tilde{y}''(t), \ldots \tilde{y}^{(n-1)}(t)\right) + \tilde{w}(t)$.

Let

$$\mathcal{Y}(t) = \tilde{y}(t), \tilde{y}'(t), \tilde{y}''(t), \ldots \tilde{y}^{(n-1)}(t), \text{ such that}$$

$$\tilde{\mathcal{Y}}(t; r) = [\underline{\mathcal{Y}}(t; r), \bar{\mathcal{Y}}(t; r)] = [\underline{y}(t; r), \underline{y}'(t; r), \ldots, \underline{y}^{(n-1)}(t; r), \bar{y}(t; r), \bar{y}'(t; r), \ldots, \bar{y}^{(n-1)}(t; r)]$$

Also we can write

$$[\tilde{f}(t, \tilde{\mathcal{Y}})]_r = [\underline{f}(t, \tilde{\mathcal{Y}}; r), \bar{f}(t, \tilde{\mathcal{Y}}; r)] \tag{3}$$

By using Zadeh extension principles as mentioned in Zadeh (2005), we have

$$\tilde{f}(t, \tilde{\mathcal{Y}}(t; r)) = [\underline{f}(t, \tilde{\mathcal{Y}}(t; r)), \underline{f}(t, \tilde{\mathcal{Y}}(t; r)),], \text{ such that}$$

$$\underline{f}(t, \tilde{\mathcal{Y}}(t; r)) = \mathcal{F}(t, \underline{\mathcal{Y}}(t; r), \bar{\mathcal{Y}}(t; r)) = \mathcal{F}(t, \tilde{\mathcal{Y}}(t; r))$$

$$\bar{f}(t, \tilde{\mathcal{Y}}(t; r)) = \mathcal{G}(t, \underline{\mathcal{Y}}(t; r), \bar{\mathcal{Y}}(t; r)) = \mathcal{G}(t, \tilde{\mathcal{Y}}(t; r))$$

Then we have

$$\underline{y}^{(n)}(t; r) = \mathcal{F}\left(t, \tilde{\mathcal{Y}}(t; r)\right) + \underline{w}(t; r) \tag{4}$$

$$\bar{y}^{(n)}(t;r) = \mathcal{G}(t, \tilde{\mathcal{Y}}(t;r)) + \bar{w}(t;r) \tag{5}$$

where the membership function of $\mathcal{F}(t, \tilde{\mathcal{Y}}(t;r)) + \underline{w}(t;r)$ and $\mathcal{G}(t, \tilde{\mathcal{Y}}(t;r)) + \bar{w}(t;r)$ can be defined as

$$\mathcal{F}(t, \tilde{\mathcal{Y}}(t;r)) + \underline{w}(t;r) = min\{\tilde{y}^{(n)}(t, \tilde{\mu}(r)) : \mu|\mu \in [\tilde{\mathcal{Y}}(t;r)]_r\}$$

$$\mathcal{G}(t, \tilde{\mathcal{Y}}(t;r)) + \bar{w}(t;r) = max\{\tilde{y}^{(n)}(t, \tilde{\mu}(r)) : \mu|\mu \in [\tilde{\mathcal{Y}}(t;r)]_r\}$$

for all $r \in [0,1]$, Eqs. (4) and (5) can be written as follows

$$\underline{\mathcal{L}}_n(\underline{y}(t;r)) - \underline{w}(t;r) - \mathcal{F}(t, \tilde{\mathcal{Y}}(t;r)) = 0 \tag{6}$$

$$\mathcal{B}\left(\underline{y}(t;r), \frac{\partial[\underline{y}]_r}{\partial t}\right) = 0 \tag{7}$$

and for the upper bound we have

$$\bar{\mathcal{L}}_n(\bar{y}(t;r)) - \bar{w}(t;r) - \mathcal{G}(t, \tilde{\mathcal{Y}}(t;r)) = 0 \tag{8}$$

$$\mathcal{B}\left(\bar{y}(t;r), \frac{\partial[\bar{y}]_r}{\partial t}\right) = 0 \tag{9}$$

According to OHAM described in Marinca et al. (2008), Eq. (1) can be written as follows:

$$\begin{aligned}&(1-p)\left[\underline{\mathcal{L}}_n([\underline{\emptyset}(t;p)]_r) - \underline{w}(t;r)\right]\\ &\quad = \underline{\mathcal{H}}(p;r)\left[\underline{\mathcal{L}}_n([\underline{\emptyset}(t;p)]_r) - \underline{w}(t;r) - \mathcal{F}\left([\tilde{\emptyset}(t;p)]_r\right)\right]\end{aligned} \tag{10}$$

$$\begin{aligned}&(1-p)\left[\bar{\mathcal{L}}_n([\bar{\emptyset}(t;p)]_r) - \bar{w}(t;r)\right]\\ &\quad = \bar{\mathcal{H}}(p;r)\left[\bar{\mathcal{L}}_n([\bar{\emptyset}(t;p)]_r) - \bar{w}(t;r) - \mathcal{G}([\tilde{\emptyset}(t;p)]_r)\right]\end{aligned} \tag{11}$$

$$\mathcal{B}\left([\tilde{\emptyset}(t;p)]_r, \frac{\partial[\tilde{\emptyset}(t;p)]_r}{\partial t}\right) = 0 \tag{12}$$

where $\underline{\mathcal{L}}_n = \frac{\partial^n[\underline{\emptyset}(t;p)]_r}{\partial t^n}$, $\bar{\mathcal{L}}_n = \frac{\partial^n[\bar{\emptyset}(t;p)]_r}{\partial t^n}$ are the linear operators of Eqs. (10) and (11) respectively, $p \in [0,1]$ is an embedding parameter, and $\tilde{\mathcal{H}}(p;r)$ is a nonzero auxiliary fuzzy function, for $p \neq 0$ and $\tilde{\mathcal{H}}(p;r) = 0$, $[\tilde{\emptyset}(t;p)]_r$ is an unknown fuzzy function, respectively. When $p = 0$ and $p = 1$, we get:

$$\begin{aligned}&[\underline{\emptyset}(t;0)]_r = \underline{y}_0(t;r), \quad [\underline{\emptyset}(t;1)]_r = \underline{y}(t;r)\\ &[\bar{\emptyset}(t;0)]_r = \bar{y}_0(t;r), \quad [\bar{\emptyset}(t;1)]_r = \bar{y}(t;r)\end{aligned} \tag{13}$$

Thus, as p increases from 0 to 1, the solution $[\tilde{\emptyset}(t;p)]_r$ varies from $\tilde{y}_0(t;r)$ to $\tilde{y}(t;r)$, where $\tilde{y}(t;r)$ is obtained from Eq. (1) for $p = 0$ we have:

$$\tilde{\mathcal{L}}_n(\tilde{y}_0(t;r)) + \tilde{w}(t;r) = 0, \quad \mathcal{B}\left(\tilde{y}_0(t;r), \frac{\partial[\tilde{y}_0]_r}{\partial t}\right) = 0 \tag{14}$$

Choose auxiliary function $\tilde{\mathcal{H}}(p;r)$ for Eqs. (3) and (4) in the form:

$$\begin{cases}\underline{\mathcal{H}}(p;r) = \underline{C}_1(r)p + \underline{C}_2(r)p^2 + \cdots = \sum_{i=1}^{n}\underline{C}_i(r)p^i\\ \underline{\mathcal{H}}(p;r) = \bar{C}_1(r)p + \bar{C}_2(r)p^2 + \cdots = \sum_{i=1}^{n}\bar{C}_i(r)p^i\end{cases} \tag{15}$$

where $C_1(r), C_2(r), \ldots$ are the constants that become function of r to be determined depending on the value of r for all. Expanding $[\tilde{\emptyset}(t;p,\check{C}_i(r))]_r$ about p, we obtain the approximate solution series:

$$[\tilde{\emptyset}(t;p,\check{C}_i(r)]_r = \tilde{y}_0(t;r) + \sum_{i=1}^{n}[\tilde{y}_i(t,\check{C}_i(r))]_r p^i \tag{16}$$

Now substitute Eq. (14) into (4) and (5), and equating the coefficients of like powers of p, the following linear equations are obtained. The zeroth order problem is given by (12), and the first and second order problems are given as follows.

First order problem:

$$\begin{cases}\underline{\mathcal{L}}_n(\underline{y}_1(t;r)) - \underline{w}(t;r) = \underline{C}_1(r)\mathcal{F}_0(\tilde{y}_0(t;r))\\ \bar{\mathcal{L}}_n(\bar{y}_1(t;r)) - \bar{w}(t;r) = \bar{C}_1(r)\mathcal{G}_0(\tilde{y}_0(t;r))\end{cases} \tag{17}$$

$$\mathcal{B}\left(\tilde{y}_1(t;r), \frac{\partial[\tilde{y}_1]_r}{\partial t}\right) = 0 \tag{18}$$

Second order problem

$$\begin{cases}\underline{\mathcal{L}}_n(\underline{y}_2(t;r)) - \underline{\mathcal{L}}_n(\underline{y}_1(t;r)) = \underline{C}_2(r)\mathcal{F}_0(\tilde{y}_0(t;r))\\ \qquad + \underline{C}_1(r)[\underline{\mathcal{L}}_n(\underline{y}_1(t;r)) + \mathcal{F}_1(\tilde{y}_0(t;r), \tilde{y}_1(t;r))]\\ \bar{\mathcal{L}}_n(\bar{y}_2(t;r)) - \bar{\mathcal{L}}_n(\bar{y}_1(t;r)) = \bar{C}_2(r)\mathcal{G}_0(\tilde{y}_0(t;r))\\ \qquad + \bar{C}_1(r)[\bar{\mathcal{L}}_n(\bar{y}_1(t;r)) + \mathcal{G}_1(\tilde{y}_0(t;r), \tilde{y}_1(t;r))]\end{cases} \tag{19}$$

$$\mathcal{B}\left(\tilde{y}_2(t;r), \frac{\partial[\tilde{y}_2]_r}{\partial t}\right) = 0 \tag{20}$$

The general nth order formula with respect to $\tilde{y}_n(t;r)$ is given by:

$$\begin{cases}\underline{\mathcal{L}}_n(\underline{y}_n(t;r)) - \underline{\mathcal{L}}_n(\underline{y}_{n-1}(t;r)) = \underline{C}_n(r)\mathcal{F}_0(\tilde{y}_0(t;r))\\ \qquad + \sum_{i=1}^{n-1}\underline{C}_i(r)\left[\underline{\mathcal{L}}_n(\underline{y}_{n-i}(t;r)) + \mathcal{F}_{n-i}\left(\sum_{j=0}^{n-1}\underline{y}_j(t;r)\right)\right]\\ \bar{\mathcal{L}}_n(\bar{y}_n(t;r)) - \bar{\mathcal{L}}_n(\bar{y}_{n-i}(t;r)) = \bar{C}_n(r)\mathcal{G}_0(\tilde{y}_0(t;r))\\ \qquad + \sum_{i=1}^{n-1}\bar{C}_i(r)\left[\bar{\mathcal{L}}_n(\bar{y}_{n-i}(t;r)) + \mathcal{G}_{n-i}\left(\sum_{j=0}^{n-1}\bar{y}_j(t;r)\right)\right]\end{cases} \tag{21}$$

$$\mathcal{B}\left(\tilde{y}_n(t;r), \frac{\partial[\tilde{y}_n]_r}{\partial t}\right) = 0 \tag{22}$$

where $\mathcal{F}\left(\sum_{j=0}^{n-1}\tilde{y}_j(t;r)\right)$ and $\mathcal{G}\left(\sum_{j=0}^{n-1}\tilde{y}_j(t;r)\right)$ are the coefficient p^n of in the expansion of $\mathcal{F}[\tilde{\emptyset}(t;p)]_r$ and $\mathcal{G}[\tilde{\emptyset}(t;p)]_r$ about the embedding parameter p

$$\begin{cases}\mathcal{F}\left(\left[\tilde{\emptyset}\left(t;p,\sum_{i=1}^{n}\tilde{C}_i(r)\right)\right]_r\right) = \mathcal{F}_0(\tilde{y}_0(t;r)) + \sum_{n=1}^{\infty}F_n\left(\sum_{i=0}^{n}[\tilde{y}_i]_r\right)p^n\\ \mathcal{G}\left([\tilde{\emptyset}\left(t;p,\sum_{i=1}^{n}\tilde{C}_i(r)\right)]_r\right) = \mathcal{G}_0(\tilde{y}_0(t;r)) + \sum_{n=1}^{\infty}\mathcal{G}_n\left(\sum_{i=0}^{n}[\tilde{y}_i]_r\right)p^n\end{cases} \tag{23}$$

The convergence of the series (10) depends upon the auxiliary fuzzy constants $\tilde{C}_1(r), \tilde{C}_2(r), \ldots$, then at $p = 1$, we obtain:

$$\tilde{y}_*\left(t, \sum_{i=1}^{n}\tilde{C}_i(r); r\right) = \tilde{y}_0(t;r) + \sum_{i=1}^{n}\left[\tilde{y}_i\left(t, \sum_{i=1}^{n}\tilde{C}_i(r)\right)\right]_r \tag{24}$$

Substituting (24) into (21) results in the following residual:

$$\begin{cases} \underline{\mathcal{R}}\left(t, \sum_{i=1}^{n} \underline{C}_i(r); r\right) = \underline{\mathcal{L}}_n\left(\underline{y}_*\left(t, \sum_{i=1}^{n} \underline{C}_i(r); r\right)\right) - \underline{w}(t; r) - F\left(\tilde{y}_*\left(t, \sum_{i=1}^{n} \tilde{C}_i(r); r\right)\right) \\ \bar{\mathcal{R}}\left(t, \sum_{i=1}^{n} \bar{C}_i(r); r\right) = \bar{\mathcal{L}}_n\left(\bar{y}_*\left(t, \sum_{i=1}^{n} \bar{C}_i(r); r\right)\right) - \bar{w}(t; r) - G\left(\tilde{y}_*\left(t, \sum_{i=1}^{n} \tilde{C}_i(r); r\right)\right) \end{cases} \tag{25}$$

If $\tilde{\mathcal{R}} = 0$, then $\tilde{y}_*$ yields the exact solution but mostly for nonlinear problems which does not happen in general. To determine the auxiliary fuzzy constants of $\tilde{C}_i(r)$, $i = 1, 2 \ldots n$, we choose t_0 and T such that optimum values of $\tilde{C}_i(r)$ for the convergent solution of the desired problem is obtained. To find the optimal values of $\tilde{C}_i(r)$ for each r-level set here, we apply the least squares method (Mabood et al., 2013a) as follows:

$$\tilde{\mathcal{S}}\left(t, \sum_{i=1}^{n} \tilde{C}_i(r); r\right) = \int_{t_0}^{T} \mathcal{R}^2\left(t, \sum_{i=1}^{n} \tilde{C}_i(r); r\right) dt \tag{26}$$

where $\tilde{\mathcal{R}}$ is the residual,

$$\begin{cases} [\underline{\mathcal{R}}]_r = \underline{\mathcal{L}}_n([\underline{y}_*]_r) - \underline{w}(t; r) - F([\tilde{y}_*]_r) \\ [\bar{\mathcal{R}}]_r = \bar{\mathcal{L}}_n([\bar{y}_*]_r) - \bar{w}(t; r) - G([\tilde{y}_*]_r) \end{cases} \tag{27}$$

and

$$\frac{\partial \tilde{\mathcal{S}}}{\partial \tilde{C}_1(r)} = \frac{\partial \tilde{\mathcal{S}}}{\partial \tilde{C}_2(r)} = \cdots \frac{\partial \tilde{\mathcal{S}}}{\partial \tilde{C}_n(r)} = 0 \tag{28}$$

where t_0 and T being the end points of Eq. (1) to locate the desired $\tilde{C}_i(r)$ $(i = 1, 2, \ldots, n)$. The convergence of the nth approximate solution depends upon unknown constants $\tilde{C}_i(r)$. With these constants known, the approximate solution (of order n) is well-determined. Thus after substituting the determined constants $\tilde{C}_i(r)$ in Eq. (24) the approximate solution of Eq. (1) can be written in the following form

$$\tilde{y}_*\left(t, \sum_{i=1}^{n} \tilde{C}_i(r); r\right) = \sum_{i=0}^{n-1} \tilde{y}_i(t; r) \tag{29}$$

It has been proved that Eq. (1) has a unique fuzzy solution in each case of r-level set for all $r \in [0, 1]$ (Mansouri and Ahmady, 2012).

4. OHAM convergence in fuzzy environments

In this section, we introduce the convergence of the solution of the nth order FIVP (1) by OHAM in Section 3. According to Theorem 2 in Gupta and Saha Ray (2014), we define the following theorem.

Theorem 4.1. Let the solution components $\tilde{y}_0(t; r)$, $\tilde{y}_1(t; r)$, $\tilde{y}_2(t; r), \ldots$ be defined as given in Eq. (14) and Eqs. (17)-(22). The series solution $\sum_{i=0}^{n-1} \tilde{y}_i(t; r)$ as defined in Eq. (29) converges if there exist $0 < \sigma < 1$ such that $\tilde{y}_{i+1}(t; r) \leqslant \sigma \tilde{y}_i(t; r)$ $\forall i \geqslant i_0$ for some $i_0 \in N$.

Proof. According to Section 2 the defuzzification of Eq. (27) for all $r \in [0, 1]$ is given by

$$\sum_{i=0}^{n-1} \tilde{y}_i(t; r) = \left[\sum_{i=0}^{n-1} \underline{y}_i(t), \sum_{i=0}^{n-1} \bar{y}_i(t)\right]_r = [\underline{y}_i(t; r), \bar{y}_i(t; r)] \tag{30}$$

As mentioned in Section 2 all fuzzy sets are subsets of $\mathbb{R}$ and the r-level sets are crisp sets, and then we can define the sequence $\{\tilde{S}_m(t; r)\}_{m=0}^{\infty} = \left[\{\underline{S}_m(t; r)\}_{m=0}^{\infty}, \{\bar{S}_m(t; r)\}_{m=0}^{\infty}\right]$ as follows.

For the lower bound of Eq. (30) we have

$$\underline{S}_0(t; r) = \underline{y}_0(t; r)$$

$$\underline{S}_1(t; r) = \underline{y}_0(t; r) + \underline{y}_1(t; r)$$

$$\underline{S}_2(t; r) = \underline{y}_0(t; r) + \underline{y}_1(t; r) + \underline{y}_2(t; r)$$

$\ldots$

$$\underline{S}_m(t; r) = \underline{y}_0(t; r) + \underline{y}_1(t; r) + \underline{y}_2(t; r) + \cdots \underline{y}_m(t; r)$$

And for the upper bound of Eq. (30) we have

$$\bar{S}_0(t; r) = \bar{y}_0(t; r)$$

$$\bar{S}_1(t; r) = \bar{y}_0(t; r) + \bar{y}_1(t; r)$$

$$\bar{S}_2(t; r) = \bar{y}_0(t; r) + \bar{y}_1(t; r) + \bar{y}_2(t; r)$$

$\ldots$

$$\bar{S}_m(t; r) = \bar{y}_0(t; r) + \bar{y}_1(t; r) + \bar{y}_2(t; r) + \cdots \bar{y}_m(t; r)$$

According to Gupta and Saha Ray (2014) we have to show that $\{\tilde{S}_m(t; r)\}_{m=0}^{\infty}$ is a Cauchy sequence in the Hilbert space $\mathbb{R}$. We start with the lower bound of Eq. (20). Consider

$$\begin{aligned} \|\underline{S}_{m+1}(t; r) - \underline{S}_m(t; r)\| &= \left\|\underline{y}_{m+1}(t; r)\right\| \leqslant \sigma \left\|\underline{y}_m(t; r)\right\| \\ &\leqslant \sigma^2 \left\|\underline{y}_{m-1}(t; r)\right\| \leqslant \cdots \leqslant \sigma^{m-i_0-1} \left\|\underline{y}_{i_0}(t; r)\right\| \end{aligned}$$

Now for each $r \in [0, 1]$, $n.m \in N$ and $m \geqslant n \geqslant i_0$

$$\begin{aligned} \|\underline{S}_m(t; r) - \underline{S}_n(t; r)\| &= \|(\underline{S}_m(t; r) - \underline{S}_{m-1}(t; r)) \\ &\quad + (\underline{S}_{m-1}(t; r) - \underline{S}_{m-2}(t; r)) + \cdots + (\underline{S}_{n+1}(t; r) - \underline{S}_n(t; r))\| \\ &\leqslant \|\underline{S}_m(t; r) - \underline{S}_{m-1}(t; r)\| + \|\underline{S}_{m-1}(t; r) - \underline{S}_{m-2}(t; r)\| + \cdots \\ &\quad + \|\underline{S}_{n+1}(t; r) - \underline{S}_n(t; r)\| \leqslant + \sigma^{m-i_0} \left\|\underline{y}_{i_0}(t; r)\right\| \\ &\quad + \sigma^{m-i_0-1} \left\|\underline{y}_{i_0}(t; r)\right\| + \cdots + \sigma^{n-i_0+1} \underline{y}_{i_0}(t; r) \\ &= \left(\frac{1 - \sigma^{m-n}}{1 - \sigma}\right) \sigma^{n-i_0+1} \underline{y}_{i_0}(t; r) \end{aligned}$$

This implies $\lim_{m,n \to \infty} \|\underline{S}_m(t; r) - \underline{S}_n(t; r)\| = 0$ (since $0 < \sigma < 1$). Therefore, $\{\underline{S}_m(t; r)\}_{m=0}^{\infty}$ is a Cauchy sequence in the Hilbert space $\mathbb{R}$.

Similarly for the upper bound of Eq. (30) we consider

$$\begin{aligned} \left\|\bar{S}_{m+1}(t; r) - \bar{S}_m(t; r)\right\| &= \|\bar{y}_{m+1}(t; r)\| \leqslant \sigma \|\bar{y}_m(t; r)\| \\ &\leqslant \sigma^2 \|\bar{y}_{m-1}(t; r)\| \leqslant \cdots \\ &\leqslant \sigma^{m-i_0-1} \|\bar{y}_{i_0}(t; r)\| \end{aligned}$$

Now for each $r \in [0, 1]$, $n.m \in N$ and $m \geqslant n \geqslant i_0$

$$
\begin{aligned}
&\left\|\bar{S}_m(t;r)-\bar{S}_n(t;r)\right\|\\
&\quad=\left\|(\bar{S}_m(t;r)-\bar{S}_{m-1}(t;r))+(\bar{S}_{m-1}(t;r)-\bar{S}_{m-2}(t;r))+\cdots\right.\\
&\qquad\left.+(\bar{S}_{n+1}(t;r)-\bar{S}_n(t;r))\right\|\\
&\quad\leqslant\left\|\bar{S}_m(t;r)-\bar{S}_{m-1}(t;r)\right\|+\left\|\bar{S}_{m-1}(t;r)-\bar{S}_{m-2}(t;r)\right\|+\cdots\\
&\qquad+\left\|\bar{S}_{n+1}(t;r)-\bar{S}_n(t;r)\right\|\\
&\quad\leqslant+\sigma^{m-i_0}\left\|\bar{y}_{i_0}(t;r)\right\|+\sigma^{m-i_0-1}\left\|\bar{y}_{i_0}(t;r)\right\|+\cdots\\
&\qquad+\sigma^{n-i_0+1}\left\|\bar{y}_{i_0}(t;r)\right\|=\left(\frac{1-\sigma^{m-n}}{1-\sigma}\right)\sigma^{n-i_0+1}\left\|\bar{y}_{i_0}(t;r)\right\|
\end{aligned}
$$

This implies $\lim_{m,n\to\infty}\|\bar{S}_m(t;r)-\bar{S}_n(t;r)\|=0$ (since $0<\sigma<1$). Therefore, $\{\bar{S}_m(t;r)\}_{m=0}^{\infty}$ is a Cauchy sequence in the Hilbert space $\mathbb{R}$. Thus $\{\tilde{S}_m(t;r)\}_{m=0}^{\infty}$ is a Cauchy sequence in the Hilbert space $\mathbb{R}$ and hence the series solution $\left[\underline{y}_i(t;r),\bar{y}_i(t;r)\right]=\sum_{i=0}^{n-1}\bar{y}_i(t;r)$ converges for each r-level set. □

5. Numerical examples

Example 1. Consider the second-order fuzzy linear differential equation (Xiaobin and Dequan, 2013)

$$y''(t)=4y'(t)-4y(t)+4t-4,\quad t\geqslant 0 \tag{31}$$

$$y(0)=(2+r,4-r),\quad y'(0)=(3+2r,9-2r)$$

$$\forall r\in[0,1]$$

The exact solution of Eq. (31) was given in Xiaobin and Dequan (2013). According to Section 3 we can construct fifth order OHAM series as follows.

Zeroth order problem:

$$i=\begin{cases}\tilde{y}_0''(t;r)=4t-4\\ \tilde{y}_0(0;r)=[2+r,4-r]\tilde{y}_0'(0;r)=[3+2r,9-2r]\end{cases} \tag{32}$$

First–fifth order problems

$$(1-p)\left(\sum_{i=1}^{5}\tilde{y}_i''(t;r)p^i\right)=\sum_{i=1}^{5}\tilde{C}_i(r)p^i\left(\left(\sum_{i=1}^{5}\tilde{y}_i''(t;r)p^i\right)-4\left(\sum_{i=1}^{5}\tilde{y}_i'(t;r)p^i\right)+4\left(\sum_{i=1}^{5}\tilde{y}_i(t;r)p^i\right)+(4-4t)\right) \tag{33}$$

$$\tilde{y}_i(0;r)=0,\quad \tilde{y}_i'(0;r)=0$$

Table 1 Optimal values of $\sum_{i=1}^{5}\underline{C}_i(r)$ given by 5-order of OHAM for Eq. (31).

r	$\underline{C}_1(r)$	$\underline{C}_2(r)$	$\underline{C}_3(r)$	$\underline{C}_4(r)$	$\underline{C}_5(r)$
0	−1.04098	0.00047659	−0.000013094	3.5393×10^{-7}	-7.19469×10^{-9}
0.2	−1.04134	0.00049515	−0.000014174	4.19711×10^{-7}	-2.53879×10^{-8}
0.4	−1.04087	0.00047063	−0.000012759	3.38551×10^{-7}	-5.52697×10^{-9}
0.6	−1.04125	0.00063290	−0.00040490	1.41883×10^{-7}	-5.33321×10^{-9}
0.8	−1.041203	0.00048767	−0.000013721	3.82523×10^{-7}	-5.43775×10^{-8}
1	−1.041131	0.00048390	−0.000013505	3.73537×10^{-7}	-7.88364×10^{-9}

Table 2 Optimal values of $\sum_{i=1}^{5}\bar{C}_i(r)$ given by 5-order of OHAM for Eq. (31).

r	$\bar{C}_1(r)$	$\bar{C}_2(r)$	$\bar{C}_3(r)$	$\bar{C}_4(r)$	$\bar{C}_5(r)$
0	−1.04094	0.00047412	−0.000012945	3.46279×10^{-7}	-6.89188×10^{-9}
0.2	−1.04067	0.00046043	−0.000012170	3.09365×10^{-7}	-5.60042×10^{-9}
0.4	−1.04035	0.00044264	−0.000011113	2.56184×10^{-7}	-3.58411×10^{-9}
0.6	−1.04030	0.00043985	−0.000010951	2.48296×10^{-7}	-3.30018×10^{-9}
0.8	−1.04144	0.00049992	−0.000014419	4.14407×10^{-7}	-1.39664×10^{-8}
1	−1.04100	0.00047708	−0.000013115	3.54585×10^{-7}	-7.19558×10^{-9}

Table 3 Comparison of the result accuracy of 5-order OHAM at $t=0.1$ and the method (Xiaobin and Dequan, 2013) at $t=0.001$ for the lower and the upper solution of Eq. (31) for all $r\in[0,1]$.

r	$[\underline{\text{Error}}]_r$ Xiaobin and Dequan (2013)	$\underline{E}(0.1;r)$ OHAM	$[\overline{\text{Error}}]_r$ Xiaobin and Dequan (2013)	$\bar{E}(0.1;r)$ OHAM
0	0.00099871222831	5.77315×10^{-15}	0.00101009737569	7.99360×10^{-15}
0.2	0.00119975860278	7.11068×10^{-10}	0.00080905100122	5.15143×10^{-14}
0.4	0.00140080497724	2.75814×10^{-11}	0.00060800462676	4.97379×10^{-14}
0.6	0.00160185135171	3.11132×10^{-10}	0.00040695825230	5.68434×10^{-14}
0.8	0.00180289772617	1.41117×10^{-9}	0.00020591187783	3.94784×10^{-10}
1	0.00200394410063	3.99680×10^{-15}	0.00000486550337	3.55271×10^{-15}

where $r \in [0,1]$ and $i = 1, 2, 3, 4, 5$. Using mathematica package to find the solutions for the lower and the upper bounds for the problems (30) and (31), we obtain

$$\tilde{y}_*(t;r) = \tilde{y}_0(t;r) + \tilde{y}_1(t, \tilde{C}_1(r); r) + \tilde{y}_2\left(t, \sum_{i=1}^{2} \tilde{C}_i(r); r\right) + \tilde{y}_3\left(t, \sum_{i=1}^{3} \tilde{C}_i(r); r\right) + \tilde{y}_4\left(t, \sum_{i=1}^{4} \tilde{C}_i(r); r\right) + \tilde{y}_5\left(t, \sum_{i=1}^{5} \tilde{C}_i(r); r\right) \quad (34)$$

By using the least square method that was mentioned in Section 3, we can compute the optimal values of $\tilde{C}_1(r)$, $\tilde{C}_2(r)$, $\tilde{C}_3(r)$, $\tilde{C}_4(r)$ and $\tilde{C}_5(r)$ as shown in $t = 0.1$ the following tables below (see Tables 1–3).

Now we can tabulate the absolute errors $[\underline{E}]_r$ and $[\bar{E}]_r$ of the approximate solutions $\underline{y}(0.1; r)$ and $\bar{y}(0.1; r)$ obtained by 5-order OHAM series solution compared with undetermined fuzzy coefficients method in (Xiaobin and Dequan, 2013) for all $r \in [0,1]$ as follows.

We can conclude from the above table the accuracy of the approximate solution of Eq. (31) solved by 5-order OHAM for all $r \in [0,1]$ when $t = 0.1$ is better than undetermined fuzzy coefficients method (Seikkala, 1987) when $t = 0.001$. The next figure shows the 5-order OHAM approximate solution $\tilde{y}(t; r)$ compared with the exact solution $\tilde{y}(t; r)$ for all $r \in [0,1]$ at: $t = 0.1$: (see Fig. 2).

Example 2. Consider the fourth order linear FIVP with triangular fuzzy initial conditions (Khodadad and Moghadam, 2006)

$$\tilde{y}^{(4)}(t) = \tilde{y}'''(t) + \tilde{y}''(t) + \tilde{y}'(t) + 2\tilde{y}(t), \quad t \in [0,1] \quad (35)$$

$$\tilde{y}(0) = \left[0.5\sqrt{r} - 0.3, 0.2\sqrt{1-r} + 0.2\right]$$

$$\tilde{y}'(0) = \left[0.4e^{r} - 0.3, 0.4e^{2-r} - 0.3\right]$$

$$\tilde{y}''(0) = \left[e^{r}, e^{2-r}\right]$$

$$\tilde{y}'''(0) = [r+2, 4-r], \quad \forall r \in [0,1]$$

The exact solution of the Eq. (35) was given in Khodadad and Moghadam (2006). Applying OHAM in Section 3 on Eq. (35) we obtain:

Zero order problem

$$\tilde{y}_0^{(4)}(t;r) = 0 \quad (36)$$

$$y_0(0;r) = \left[\frac{\sqrt{r}}{2} - 0.3, 0.2\sqrt{1-r} + 0.2\right],$$

$$y_0'(0;r) = \left[0.4e^{r} - 0.3, 0.4e^{2-r} - 0.3\right]$$

$$y_0''(0;r) = \left[e^{r}, e^{2-r}\right], \quad y_0'''(t;r) = [r+2, 4-r]$$

First to sixth order problem

$$\left\{(1-p)\left(\sum_{i=1}^{6} p^i \tilde{y}_i^{(4)}(t;r)\right)\right\} - pC_1\left\{\left(\sum_{i=0}^{6} p^i \tilde{y}_i^{(4)}(t;r)\right) - \left(\sum_{i=0}^{6} p^i \tilde{y}_i'''(t;r)\right) - \left(\sum_{i=0}^{6} p^i \tilde{y}_i''(t;r)\right) - \left(\sum_{i=0}^{6} p^i \tilde{y}_i'(t;r)\right) - 2\left(\sum_{i=0}^{6} p^i \tilde{y}_i(t;r)\right)\right\} \quad (37)$$

$$\tilde{y}_i(0;r) = 0, \quad \tilde{y}_i'(0;r) = 0, \quad \tilde{y}_i''(0;r) = 0, \quad \tilde{y}_i'''(0;r) = 0$$

where $r \in [0,1]$ and $i = 1, 2, 3, 4, 5, 6$.

Using mathematica package to find the solutions for the lower and the upper bounds for the problems (36) and (35), we obtain (see Fig. 3)

$$\tilde{y}_*(t;r) = \tilde{y}_0(t;r) + \sum_{i=1}^{6} \tilde{y}_1(t, \tilde{C}_1(r); r) \quad (38)$$

By using the least square method that was mentioned in Section 3, we can compute the optimal values of $\tilde{C}_1(r)$ as shown in the following tables below (see Table 4).

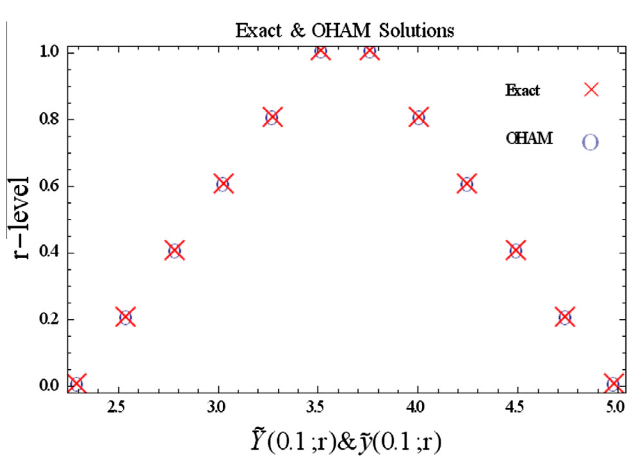

Figure 2 Approximate 5-order OHAM and the exact solution of Eq. (31) at $t = 0.1$.

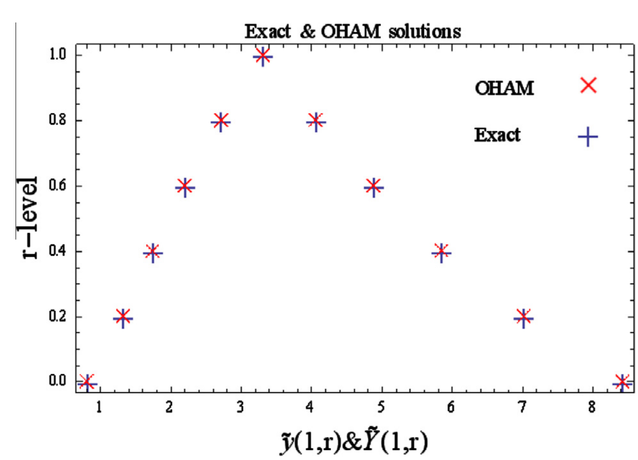

Figure 3 Approximate 6-order OHAM and exact solutions of Eq. (35) at $t = 1$.

Table 4 Comparison of the result accuracy solved by 6-order of OHAM and the exact solution at $t = 1$ for the lower and the upper solution of Eq. (35) for all $r \in [0, 1]$.

r	$\underline{C}_1(r)$	$\underline{E}(1;r)$ OHAM	$\bar{C}_1(r)$	$\bar{C}_1(r)$
0	−1.103529525572901	1.54824×10^{-8}	−1.103736577943245	−1.103736577943245
0.2	−1.103744444421023	2.11509×10^{-8}	−1.1037575787033083	−1.1037575787033083
0.4	−1.103792071040843	2.57572×10^{-8}	−1.1037732976880386	−1.1037732976880386
0.6	−1.1038080533565133	3.06465×10^{-8}	−1.103794588365459	−1.103794588365459
0.8	−1.10381293474532	3.60336×10^{-8}	−1.1038103704152256	−1.1038103704152256
1	−1.1038081037895575	4.21204×10^{-8}	−1.1038081037895575	−1.1038081037895575

6. Conclusions

In this paper, we studied and applied the optimal homotopy asymptotic method in finding solution of high order fuzzy initial value problems involving linear ordinary differential equations. To the best of our knowledge, this is the first attempt for solving the high order FIVPs with OHAM. The method has been formulated to obtain an approximate solution of general high order FIVP. The convergence theorem of OHAM for solving FIVPs has been presented and proved. In OHAM, the control and adjustment of the convergence of the series solution using the convergence control parameters are achieved in a simple way in a numerical example including second order linear and fourth order fuzzy initial value problems showing the capability and efficiency of the OHAM. We obtained accurate results by using even low order approximation. Moreover, this technique converges to the exact solution and requires less computational work directly without reducing to first order system. The numerical results obtained by OHAM satisfy the fuzzy number properties by taking the convex fuzzy number shape. Moreover the procedure of OHAM has advantages over some existing analytical approximation methods.

References

Abbasbandy, S., Allahviranloo, T., Darabi, P., Sedaghatfar, O., 2011. Variational iteration method for solving *n*th order fuzzy differential equations. Math. Comput. Appl. 16 (4), 819–829.

Abbod, F.M., Von Keyserlingk, D.G., Linkens, D.A., Mahfouf, M., 2001. Survey of utilization of fuzzy technology in medicine and healthcare. Fuzzy Sets Syst. 120, 331–3491.

Ahmad, M.Z., De Baets, B., 2009. A predator–prey model with fuzzy initial populations. IFSA-EUSFLAT.

Allahviranlo, T., Khezerloo, S., Mohammadzaki, M., 2008. Numerical solution for differential inclusion by adomian decomposition method. J. Appl. Math. 5 (17), 51–62.

Alomari, A.K., Anakira, N.R., Bataineh, A.S., Hashim, I., 2013a. Approximate solution of nonlinear system of BVP arising in fluid flow problem. Math. Prob. Eng. 2013, 7. Article ID 136043.

Alomari, A.K., Awawdeh, F., Tahat, N., Ahmad, F.B., Shatanawi, W., 2013b. Multiple solutions for fractional differential equations: analytic approach. Appl. Math. Comput. 219 (17), 8893–8903.

Anakira, N.R., Alomari, A.K., Hashim, I., 2013. Optimal homotopy asymptotic method for solving delay differential equations. Math. Prob. Eng. 2013, 11. Article ID 498902.

Babolian, E., Sadeghi, H., Javadi, S., 2004. Numerically solution of fuzzy differential equations by adomian method. Appl. Math. Comput. 149, 547–557.

Barro, S., Marin, R., 2002. Fuzzy Logic in Medicine. Physica-Verlag, Heidelberg.

Bodjanova, S., 2006. Median alpha-levels of a fuzzy number. Fuzzy Sets Syst. 157 (7), 879–891.

Dubois, D., Prade, H., 1982. Towards fuzzy differential calculus, Part 3: differentiation. Fuzzy Sets Syst. 8, 225–233.

El Naschie, M.S., 2005. From experimental quantum optics to quantum gravity via a fuzzy kahler manifold. Chaos Solitons Fract. 25, 969–977.

Fard, O.S., 2009. An iterative scheme for the solution of generalized system of linear fuzzy differential equations. World Appl. Sci. J. 7, 1597–11604.

Ghanbari, M., 2009. Numerical solution of fuzzy initial value problems under generalization differentiability by HPM. Int. J. Ind. Math. 1 (1), 19–39.

Gupta, A.K., Saha Ray, S., 2014a. Comparison between homotopy perturbation method and optimal homotopy asymptotic method for the solution of Boussinesq–Burgers equations. Comput. Fluids 103 (2014), 34–41.

Gupta, A.K., Saha Ray, S., 2014b. On the solutions of fractional burgers-fisher and generalized fisher's equations using two reliable methods. Int. J. Math. Math. Sci. 2014, 1–16.

Herisanu, N., Marinca, V., Dordea, T., Madescu, G., 2008. A new analytical approach to nonlinear vibration of an electric machine. Proc. Romanian Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci. 9 (3), 229–236.

Herisanu, N., Marinca, V., Madescu, Gh., 2015. An analytical approach to non-linear dynamical model of a permanent magnet synchronous generator. Wing Energy. http://dx.doi.org/10.1002/we.1785.

Iqbal, S., Idrees, M., Siddiqui, A., Ansari, A., 2010. Some solutions of the linear and nonlinear Klein–Gordon equations using the optimal homotopy asymptotic method. Appl. Math. Comput. 216 (1), 2898–2909, 21.

Kaleva, O., 1987. Fuzzy differential equations. Fuzzy Sets Syst. 24, 301–317.

Khodadad, M., Moghadam, M., 2006. Linear k-step methods for solving *n*th-order fuzzy differential equations. Adv. Fuzzy Math. 1 (1), 23–34.

Mabood, F., 2014. Comparison of optimal homotopy asymptotic method and homotopy perturbation method for strongly nonlinear equation. J. Assoc. Arab Univ. Basic Appl. Sci. 16, 21–26.

Mabood, F., Ismail, A.I.M., Hashim, I., 2013a. Application of optimal homotopy asymptotic method for the approximate solution of Riccati equation. Sains Malaysiana 42 (6), 863–867.

Mabood, F., Khan, W.A., Ismail, A.I.M., 2013b. Optimal homotopy asymptotic method for flow and heat transfer of a viscoelastic fluid in an axisymmetric channel with a porous wall. PLoS One 8 (12), e83581. http://dx.doi.org/10.1371/journal/pone.0083581.

Mabood, F., Khan, W.A., Ismail, A.I.M., 2014a. Optimal homotopy asymptotic method for heat transfer in hollow sphere with robin boundary condition. Heat Transfer Asian Res. 42 (2), 124–133.

Mabood, F., Khan, W.A., Ismail, A.I.M., 2014b. Solution of nonlinear boundary layer equation for flat plate via optimal homotopy asymptotic method. Heat Transfer Asian Res. 43 (3), 197–203.

Mansouri, S., Ahmady, S.N., 2012. A numerical method for solving *n*th-order fuzzy differential equation by using characterization theorem. Commun. Numer. Anal., 12 Article ID cna-00054.

Marinca, V., Herisanu, N., 2008. Application of optimal homotopy asymptotic method for solving nonlinear equations arising in heat transfer. Int. Commun. Heat Mass Transfer 35, 710–715.

Marinca, V., Herisanu, N., 2014. The optimal homotopy asymptotic method for solving Blasius equation. Appl. Math. Comput. 231, 134–139.

Marinca, V., Herisanu, N., Nemes, I., 2008. An optimal homotopy asymptotic method with application to thin film flow. Central Eur. J. Phys. 6 (3), 648–653.

Marinca, V., Herisanu, N., Bota, C., Marinca, B., 2009. An optimal homotopy asymptotic method applied to steady flow of a fourth-grade fluid past a porous plate. Appl. Math. Lett. 22, 245–251.

Omer, A., Omer, O., 2013. A pray and predator model with fuzzy initial values. Hacettepe J. Math. Stat. 41 (3), 387–395.

Sahu, P.K., Saha Ray, S., 2015. Two-dimensional Legendre wavelet method for the numerical solutions of fuzzy integro-differential equations. J. Intell. Fuzzy Syst. 28 (2015), 1271–1279.

Salahshour, S., 2011. Nth-order fuzzy differential equations under generalized differentiability. J. Fuzzy Set Value Anal., 14 Article ID jfsva-00043.

Seikkala, S., 1987. On the fuzzy initial value problem. Fuzzy Sets Syst. 24 (3), 319–330.

Xiaobin, G., Dequan, S., 2013. Approximate solution of *n*th-order fuzzy linear differential equations. Math. Prob. Eng., 12

Zadeh, L.A., 1965. Fuzzy Sets Inf. Control 8, 338–353.

Zadeh, L.A., 2005. Toward a generalized theory of uncertainty. Inf. Sci. 172 (1–2), 1–40.