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## تعميم إضافي لطريقة مفكوك (G'/G) العامة لحل معادلة تطور لا خطي

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### الملخص:

في هذا البحث يتم تعميم إضافي آخر لطريقة مفكوك (G'/G) لإيجاد حلول موجية مسافرة جديدة. بالإضافة يتم الاستعانة بمعادلة تفاضلية لا خطية كمعادلة مساعدة جديدة ذات عدة بارامترات. تم تطبيق الطريقة لمعادلة (ZKBBM) لاستنتاج حلول عامة، والتي تؤول إلى حلول سابقة معروفة لقيم معينة لبعض البارامترات.



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ORIGINAL ARTICLE

# Further extension of the generalized and improved ( $G'/G$ )-expansion method for nonlinear evolution equation



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## KEYWORDS

Further extension of the generalized and improved ( $G'/G$ )-expansion method; Nonlinear ordinary differential equation; ZKBBM equation; Travelling wave solutions

**Abstract** In this article, the generalized and improved ( $G'/G$ )-expansion method has been proposed for further extension to generate many new travelling wave solutions. In addition, nonlinear ordinary differential equation is implemented as auxiliary equation including many parameters instead of linear ordinary differential equation. Moreover, the presentation of the travelling wave solutions is quite new. The effectiveness and reliability of the method are shown by its application to the Zakharov–Kuznetsov–Benjamin–Bona–Mahony (ZKBBM) equation. Some of our generated solutions turned into some known solutions, when parameters consider specific values and others are new.

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## 1. Introduction

Nonlinear partial differential equations (PDEs) play a vital role in the field of science and engineering, such as fluid mechanics, plasma physics, solid state physics, chemical physics, quantum mechanics, optical fibres, electricity, geochemistry, meteorology and many others (Kaya, 2004; Wazwaz, 2005; Wang et al., 2006; Elhanbaly and Abdou, 2007; Bekir, 2008; Shang, 2010; Bekir and Uygun, 2012). Due to important

applications of nonlinear PDEs in real world problems, it is required to generate their analytical solutions. With the help of analytical solutions, if exist, the phenomena modelled can be better understood by these nonlinear PDEs. In the past several decades, many powerful methods have been developed by a diverse group of researchers to construct exact solutions. For example, the Cole–Hopf transformation method (Cole, 1951; Hopf, 1950), the Hirota's bilinear transformation method (Hirota, 1971), the truncated Painleve expansion method (Weiss et al., 1982), the Backlund transformation method (Rogers and Shadwick, 1982), the Weierstrass elliptic function method (Kudryashov, 1990), the inverse scattering method (Ablowitz and Clarkson, 1991), the tanh method (Malfliet, 1992), the tanh–coth method (Wazwaz, 2007), the Riccati equation method (Yan and Zhang, 2001; Naher and Abdullah, 2012a,b; Naher et al., 2013b), the Jacobi elliptic

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function expansion method (Liu et al., 2001), the F-expansion method (Wang and Li, 2005; Abdou, 2007), the Exp-function method (He and Wu, 2006; Naher et al., 2012a; Mohyud-Din et al., 2010; Ma and Zhu, 2012) and others (Bagarti et al., 2012; Belgacem et al., 2013; Bibi and Mohyud-Din, 2014; Dai and Zhu, 2013; Noor et al., 2013; Zhu, 2013; Dai and Zhu, 2014; Dai et al., 2014; Zhu and Pan, 2014). However, basically there is no unified method that can be applied to handle all types of nonlinear evolution equations (NLEEs).

Recently, Wang et al. (2008) presented a direct method and called the ( $G'/G$ )-expansion method to generate travelling wave solutions of some NLEEs. Many scientists effectively implemented the ( $G'/G$ )-expansion method to solve various kinds of nonlinear problems for obtaining travelling wave solutions (Zayed and Gepreel, 2009; Feng et al., 2011; Naher et al., 2011; Naher and Abdullah, 2012c; Bekir and Aksoy, 2012; Alzaidy, 2013).

The efficiency of the ( $G'/G$ )-expansion method has been demonstrated through its extension and improvement. Such as, Zayed (2009) proposed extended ( $G'/G$ )-expansion method, where  $G(\xi)$  satisfies the Jacobi elliptic equation. Zhang et al. (2010) improved the ( $G'/G$ )-expansion method. After that many scientists applied the improved ( $G'/G$ )-expansion method for generating many travelling wave solutions for nonlinear PDEs, (Hamad et al., 2011; Naher and Abdullah, 2012d,e,f; Naher et al., 2012b; Ye and Cai, 2011).

Lately, Guo and Zhou (2010) introduced the extended ( $G'/G$ )-expansion method for the Whitham–Broer–Kaup-like equation and couple Hirota–Satsuma KdV equations. Then, Zayed and Al-Joudi (2010) implemented this method whilst Zayed and El-Malky (2011) solved higher-dimensional evolution equations by using this method. Later on, Akbar et al. (2012) proposed another extension of the ( $G'/G$ )-expansion method to the KdV equation, the ZKBBM equation and the strain wave equation in microstructured solids, called the generalized and improved ( $G'/G$ )-expansion method. Additional parameter is added in the presentation of the solutions, but the same second-order linear ordinary differential equation has been used as auxiliary equation. Afterward, Naher et al. (2013a) studied higher dimensional nonlinear PDE via this method and so on. In a whilst, Liu et al. (2012) introduced another approach of ( $G'/G$ ) method which is also the improved ( $G'/G$ )-expansion method.

Very recently, Naher and Abdullah (2013) presented a new approach of the ( $G'/G$ )-expansion method and a new approach of the generalized ( $G'/G$ )-expansion method. In this method, nonlinear ODE has been executed as auxiliary equation. On the other hand, the presentations of the travelling wave solutions are quite different.

In the present work, we improve the ( $G'/G$ )-expansion method called further extension of the generalized and improved ( $G'/G$ )-expansion method. In the method, nonlinear ODE is used as auxiliary equation with many parameters. It is quite interesting to point out that, the sign of the parameters can take the opportunity to motivate the types of travelling wave solutions. For illustration and to depict the advantages of the proposed method, the ZKBBM equation has been studied and generated abundant and more types of new travelling wave solutions.

## 2. Description of new extension of the generalized and improved ( $G'/G$ )-expansion method

Let us consider a general nonlinear PDE:

$$F(u, u_t, u_x, u_{t^2}, u_{xt}, u_{xx}, \dots) = 0, \tag{1}$$

where  $u = u(x,t)$  is an unknown function,  $F$  is a polynomial in  $u(x,t)$  and its partial derivatives in which the highest order partial derivatives and nonlinear terms are involved. The main steps of the method are as follows:

*Step 1.* We suppose that the combination of real variables  $x$  and  $t$  by a complex variable  $\varphi$

$$u(x, t) = u(\varphi), \quad \varphi = x \pm W t, \tag{2}$$

where  $W$  is the speed of the travelling wave. Now using Eqs. (2), (1) is converted into an ordinary differential equation for  $u = u(\varphi)$ :

$$Q(u, u', u'', u''', \dots) = 0, \tag{3}$$

where the superscripts indicate the ordinary derivatives with respect to  $\varphi$ .

*Step 2.* According to the possibility, Eq. (3) can be integrated term by term one or more times, yields constant(s) of integration. The integral constant may be zero, for simplicity.

*Step 3.* Suppose that the travelling wave solution of Eq. (3) can be expressed as follows:

$$u(\varphi) = \sum_{j=-N}^N \alpha_j (d + H)^j + \sum_{j=1}^N \frac{\beta_j}{(d + H)^j}, \tag{4}$$

where either  $\alpha_{-N}$  or  $\alpha_N$  or  $\beta_N$  may be zero, but these  $\alpha_{-N}, \alpha_N$  and  $\beta_N$  cannot be zero at a time,  $\alpha_j (j = 0, \pm 1, \pm 2, \dots, \pm N), \beta_j (j = 1, 2, \dots, N)$  and  $d$  are arbitrary constants to be determined later and  $H(\varphi)$  is

$$H(\varphi) = (G'/G), \tag{5}$$

where  $G = G(\varphi)$  satisfies the following nonlinear ordinary differential equation (ODE)

$$AGG'' - BGG' - C(G')^2 - EG^2 = 0, \tag{6}$$

where the primes denote derivatives with respect to  $\varphi$  and  $A, B, C$ , and  $E$  are real parameters.

*Step 4.* To determine the positive integer  $N$ , taking the homogeneous balance between the highest order nonlinear terms and the highest order derivatives appearing in Eq. (3).

*Step 5.* Substituting Eqs. (4) and (6) including Eq. (5) into Eq. (3) with the value of  $N$  obtained in Step 4 we obtain polynomials in  $(d + H)^N (N = 0, \pm 1, \pm 2, \dots)$  and  $(d + H)^{-N} (N = 1, 2, 3, \dots)$ . Then, we collect each coefficient of the resulted polynomials to zero, yields a set of algebraic equations for  $\alpha_j (j = 0, \pm 1, \pm 2, \dots, \pm N), \beta_j (j = 1, 2, \dots, N), d$  and  $W$ .

*Step 6.* Suppose that the value of the constants  $\alpha_j (j = 0, \pm 1, \pm 2, \dots, \pm N), \beta_j (j = 1, 2, \dots, N), d$  and  $W$  can be found by solving the algebraic equations which are obtained in step 5. Since the general

solution of Eq. (6) is well known to us, substituting the values of constants into Eq. (4), we can obtain more general type and many new travelling wave solutions of the nonlinear partial differential Eq. (1).

Using the general solution of Eq. (6), we have the following solutions of Eq. (5):

**Family 1.** When  $B \neq 0, \Psi = A - C$  and  $\Omega = B^2 + 4E(A - C) > 0$ ,

$$H(\varphi) = \left( \frac{G'}{G} \right) = \frac{B}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi} \frac{C_1 \sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\varphi\right) + C_2 \cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\varphi\right)}{C_1 \cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\varphi\right) + C_2 \sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\varphi\right)} \quad (7)$$

**Family 2.** When  $B \neq 0, \Psi = A - C$  and  $\Omega = B^2 + 4E(A - C) < 0$ ,

$$H(\varphi) = \left( \frac{G'}{G} \right) = \frac{B}{2\Psi} + \frac{\sqrt{-\Omega}}{2\Psi} \frac{-C_1 \sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\varphi\right) + C_2 \cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\varphi\right)}{C_1 \cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\varphi\right) + C_2 \sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\varphi\right)} \quad (8)$$

**Family 3.** When  $B \neq 0, \Psi = A - C$  and  $\Omega = B^2 + 4E(A - C) = 0$ ,

$$H(\varphi) = \left( \frac{G'}{G} \right) = \frac{B}{2\Psi} + \frac{C_2}{C_1 + C_2\varphi} \quad (9)$$

**Family 4.** When  $B = 0, \Psi = A - C$  and  $\Delta = \Psi E > 0$ ,

$$H(\varphi) = \left( \frac{G'}{G} \right) = \frac{\sqrt{\Delta}}{\Psi} \frac{C_1 \sinh\left(\frac{\sqrt{\Delta}}{\Psi}\varphi\right) + C_2 \cosh\left(\frac{\sqrt{\Delta}}{\Psi}\varphi\right)}{C_1 \cosh\left(\frac{\sqrt{\Delta}}{\Psi}\varphi\right) + C_2 \sinh\left(\frac{\sqrt{\Delta}}{\Psi}\varphi\right)} \quad (10)$$

**Family 5.** When  $B = 0, \Psi = A - C$  and  $\Delta = \Psi E < 0$ ,

$$H(\varphi) = \left( \frac{G'}{G} \right) = \frac{\sqrt{-\Delta}}{\Psi} \frac{-C_1 \sin\left(\frac{\sqrt{-\Delta}}{\Psi}\varphi\right) + C_2 \cos\left(\frac{\sqrt{-\Delta}}{\Psi}\varphi\right)}{C_1 \cos\left(\frac{\sqrt{-\Delta}}{\Psi}\varphi\right) + C_2 \sin\left(\frac{\sqrt{-\Delta}}{\Psi}\varphi\right)} \quad (11)$$

### 3. Application of the method

In this section, the ZKBBM equation has been investigated by applying the proposed method to construct a rich class of new travelling wave solutions.

#### 3.1. The ZKBBM equation

Let us consider the ZKBBM equation

$$u_t + u_x - 2auu_x - bu_{xx} = 0. \quad (12)$$

Now, we use the wave transformation  $\varphi = x + Wt$  into the Eq. (12), which yields:

$$(1 + W)u' - 2auu' - bVu''' = 0. \quad (13)$$

Eq. (13) is integrable, therefore, integrating with respect to  $\varphi$  once yields:

$$K + (1 + W)u - au^2 - bVu'' = 0, \quad (14)$$

where  $K$  is an integral constant which is to be determined.

Taking homogeneous balance between  $u^2$  and  $u''$  in Eq. (14), we obtain  $N = 2$ .

Therefore, the solution of Eq. (14) is of the form:

$$v(\varphi) = \alpha_0 + \alpha_1(d + H) + \alpha_2(d + H)^2 + (\alpha_{-1} + \beta_1)(d + H)^{-1} + (\alpha_{-2} + \beta_2)(d + H)^{-2}, \quad (15)$$

where  $\alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2$  and  $d$  are constants to be determined.

Substituting Eq. (15) together with Eqs. (5) and (6) into Eq. (14), the left-hand side is converted into polynomials in  $(d + H)^N$  ( $N = 0, \pm 1, \pm 2, \dots$ ) and  $(d + H)^{-N}$  ( $N = 1, 2, 3, \dots$ ). We collect each coefficient of these resulted polynomials to zero, yields a set of simultaneous algebraic equations (for simplicity, which are not presented) for  $\alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, d, K$  and  $W$ . Solving these algebraic equations with the help of algebraic software Maple, we obtain the following.

*Case 1:*

$$\begin{aligned} \alpha_{-2} &= -\beta_2, \quad \alpha_{-1} = -\beta_1, \\ \alpha_0 &= \frac{A^2 + W(A^2 - bB^2) - 4bW\Psi\{3d(B + d\Psi) - 2E\}}{2aA^2}, \quad d = d, \\ \alpha_1 &= \frac{6bW\{\Psi(B + 2d\Psi)\}}{aA^2}, \quad \alpha_2 = \frac{-6bW\Psi^2}{aA^2}, \quad W = W, \\ K &= \frac{\{bW(4\Delta + B^2)\}^2 - \{A^2(1 + W)\}^2}{4aA^4}, \end{aligned} \quad (16)$$

where  $\Psi = A - C, \Delta = \Psi E, \beta_1, \beta_2, A, B, C$  and  $E$  are free parameters.

*Case 2:*

$$\begin{aligned} \alpha_{-2} &= \frac{-6bW\{d^2\Psi(d^2\Psi + 2(dB - E)) + (dB - E)^2\}}{aA^2} - \beta_2, \\ \alpha_1 &= 0, \quad \alpha_2 = 0, \\ \alpha_{-1} &= \frac{6bW\{d(2d^2\Psi^2 + \Psi(3dB - 2E)) + B(dB - E)\}}{aA^2} - \beta_1, \\ d &= d, \quad W = W, \\ \alpha_0 &= \frac{A^2 + W(A^2 - bB^2) - 4bW\Psi\{3d(B + d\Psi) - 2E\}}{2aA^2}, \\ K &= \frac{\{bW(4\Delta + B^2)\}^2 - \{A^2(1 + W)\}^2}{4aA^4}, \end{aligned} \quad (17)$$

where  $\Psi = A - C, \Delta = \Psi E, \beta_1, \beta_2, A, B, C$  and  $E$  are free parameters.

*Case 3:*

$$\begin{aligned} \alpha_{-2} &= \frac{-3bW\{8\Delta(2\Delta + B^2) + B^4\}}{8aA^2\Psi^2} - \beta_2, \\ \alpha_{-1} = -\beta_1, \alpha_0 &= \frac{2bW(4\Delta + B^2) + A^2(1 + W)}{2aA^2}, \quad \alpha_1 = 0, \\ \alpha_2 &= \frac{-6bW\Psi^2}{aA^2}, \quad d = \frac{-B}{2\Psi}, \\ K &= \frac{128b^2W^2\Delta(2\Delta + B^2) + (4bB^2W)^2 - \{A^2(1 + W)\}^2}{4aA^4}, \\ W &= W, \end{aligned} \tag{18}$$

where  $\Psi = A - C$ ,  $\Delta = \Psi E$ ,  $\beta_1, \beta_2, A, B, C$  and  $E$  are free parameters.

Case 4:

$$\begin{aligned} \alpha_{-2} &= \frac{-3bW\{8\Delta(2\Delta + B^2) + B^4\}}{8aA^2\Psi^2} - \beta_2, \quad \alpha_{-1} = -\beta_1, \\ \alpha_0 &= \frac{2bW(4\Delta + B^2) + A^2(1 + W)}{2aA^2}, \\ \alpha_1 = 0, \quad \alpha_2 = 0, d &= \frac{-B}{2\Psi}, \\ K &= \frac{\{bW(4\Delta + B^2)\}^2 - \{A^2(1 + W)\}^2}{4aA^4}, \quad W = W, \end{aligned} \tag{19}$$

where  $\Psi = A - C, \Delta = \Psi E, \beta_1, \beta_2, A, B, C$  and  $E$  are free parameters.

For Case 1: substituting Eq. (16) into Eq. (15), along with Eq. (7) and simplifying, yields the following travelling wave solutions (if  $C_1 = 0$  but  $C_2 \neq 0$ ;  $C_2 = 0$  but  $C_1 \neq 0$ ) respectively:

$$\begin{aligned} v_{11}(x, t) &= \frac{1+W}{2a} - \frac{bB^2W}{2aA^2} + \frac{bW}{aA^2} \left\{ 4\Delta + \frac{3}{2} \left( B^2 - \Omega \coth^2 \left( \frac{\sqrt{\Omega}}{2\Psi} \varphi \right) \right) \right\}, \\ v_{12}(x, t) &= \frac{1+W}{2a} - \frac{bB^2W}{2aA^2} + \frac{bW}{aA^2} \left\{ 4\Delta + \frac{3}{2} \left( B^2 - \Omega \tanh^2 \left( \frac{\sqrt{\Omega}}{2\Psi} \varphi \right) \right) \right\}, \end{aligned}$$

substituting Eq. (16) into Eq. (15), along with Eq. (8) and simplifying, our exact solutions become (if  $C_1 = 0$  but  $C_2 \neq 0$ ;  $C_2 = 0$  but  $C_1 \neq 0$ ) respectively:

$$\begin{aligned} v_{13}(x, t) &= \frac{1+W}{2a} - \frac{bB^2W}{2aA^2} + \frac{bW}{aA^2} \left\{ 4E\Psi + \frac{3}{2} \left( B^2 + \Omega \cot^2 \left( \frac{\sqrt{-\Omega}}{2\Psi} \varphi \right) \right) \right\}, \\ v_{14}(x, t) &= \frac{1+W}{2a} - \frac{bB^2W}{2aA^2} + \frac{bW}{aA^2} \left\{ 4\Delta + \frac{3}{2} \left( B^2 + \Omega \tan^2 \left( \frac{\sqrt{-\Omega}}{2\Psi} \varphi \right) \right) \right\}, \end{aligned}$$

substituting Eq. (16) into Eq. (15), together with Eq. (9) and simplifying, our obtained solution becomes:

$$v_{15}(x, t) = \frac{1+W}{2a} - \frac{bB^2W}{2aA^2} + \frac{bW}{aA^2} \left( 4\Delta + \frac{3}{2} \left( B^2 - \left( \frac{2\Psi C_2}{C_1 + C_2 \varphi} \right)^2 \right) \right),$$

similarly, substituting Eq. (16) into Eq. (15), along with Eq. (10) and simplifying, we obtain the following travelling wave solutions (if  $C_1 = 0$  but  $C_2 \neq 0$ ;  $C_2 = 0$  but  $C_1 \neq 0$ ) respectively:

$$\begin{aligned} v_{16}(x, t) &= \frac{1+W}{2a} - \frac{bB^2W}{2aA^2} + \frac{2bW}{aA^2} \left\{ 2\Delta + 3\sqrt{\Delta} \left( B \coth \left( \frac{\sqrt{\Delta}}{\Psi} \varphi \right) - \sqrt{\Delta} \coth^2 \left( \frac{\sqrt{\Delta}}{\Psi} \varphi \right) \right) \right\}, \\ v_{17}(x, t) &= \frac{1+W}{2a} - \frac{bB^2W}{2aA^2} + \frac{2bW}{aA^2} \left\{ 2\Delta + 3\sqrt{\Delta} \left( B \tanh \left( \frac{\sqrt{\Delta}}{\Psi} \varphi \right) - \sqrt{\Delta} \tanh^2 \left( \frac{\sqrt{\Delta}}{\Psi} \varphi \right) \right) \right\}, \end{aligned}$$

substituting Eq. (16) into Eq. (15), together with Eq. (11) and simplifying, our obtained exact solutions become (if  $C_1 = 0$  but  $C_2 \neq 0$ ;  $C_2 = 0$  but  $C_1 \neq 0$ ) respectively:

$$\begin{aligned} v_{18}(x, t) &= \frac{1+W}{2a} - \frac{bB^2W}{2aA^2} + \frac{2bW}{aA^2} \left\{ 2\Delta + 3\sqrt{\Delta} \left( iB \cot \left( \frac{\sqrt{-\Delta}}{\Psi} \varphi \right) + \sqrt{\Delta} \cot^2 \left( \frac{\sqrt{-\Delta}}{\Psi} \varphi \right) \right) \right\}, \\ v_{19}(x, t) &= \frac{1+W}{2a} - \frac{bB^2W}{2aA^2} + \frac{2bW}{aA^2} \left\{ 2\Delta + 3\sqrt{\Delta} \left( iB \tan \left( \frac{\sqrt{-\Delta}}{\Psi} \varphi \right) + \sqrt{\Delta} \tan^2 \left( \frac{\sqrt{-\Delta}}{\Psi} \varphi \right) \right) \right\}, \end{aligned}$$

where  $\varphi = x + Wt$ .

For Case 2: similarly, substituting Eq. (17) into Eq. (15), along with Eqs. (7)–(11) and simplifying, our travelling wave solutions become (if  $C_1 = 0$  but  $C_2 \neq 0$ ;  $C_2 = 0$  but  $C_1 \neq 0$ , for 1<sup>st</sup> two solutions, again these conditions for  $v_{23}$  and  $v_{24}$ , also same conditions could be applied for solutions  $v_{26}$  and  $v_{27}$ , moreover, mentioned conditions are implemented to solutions  $v_{28}$  and  $v_{29}$ ) respectively:

$$\begin{aligned} v_{21}(x, t) &= \alpha_0 + \alpha_{-1} \left( d + \frac{B}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi} \coth \left( \frac{\sqrt{\Omega}}{2\Psi} \varphi \right) \right)^{-1} \\ &\quad + \alpha_{-2} \left( d + \frac{B}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi} \coth \left( \frac{\sqrt{\Omega}}{2\Psi} \varphi \right) \right)^{-2}, \\ v_{22}(x, t) &= \alpha_0 + \alpha_{-1} \left( d + \frac{B}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi} \tanh \left( \frac{\sqrt{\Omega}}{2\Psi} \varphi \right) \right)^{-1} \\ &\quad + \alpha_{-2} \left( d + \frac{B}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi} \tanh \left( \frac{\sqrt{\Omega}}{2\Psi} \varphi \right) \right)^{-2}, \\ v_{23}(x, t) &= \alpha_0 + \alpha_{-1} \left( d + \frac{B}{2\Psi} + \frac{\sqrt{-\Omega}}{2\Psi} \cot \left( \frac{\sqrt{-\Omega}}{2\Psi} \varphi \right) \right)^{-1} \\ &\quad + \alpha_{-2} \left( d + \frac{B}{2\Psi} + \frac{\sqrt{-\Omega}}{2\Psi} \cot \left( \frac{\sqrt{-\Omega}}{2\Psi} \varphi \right) \right)^{-2}, \\ v_{24}(x, t) &= \alpha_0 + \alpha_{-1} \left( d + \frac{B}{2\Psi} - \frac{\sqrt{-\Omega}}{2\Psi} \tan \left( \frac{\sqrt{-\Omega}}{2\Psi} \varphi \right) \right)^{-1} \\ &\quad + \alpha_{-2} \left( d + \frac{B}{2\Psi} - \frac{\sqrt{-\Omega}}{2\Psi} \tan \left( \frac{\sqrt{-\Omega}}{2\Psi} \varphi \right) \right)^{-2}, \\ v_{25}(x, t) &= \alpha_0 + \alpha_{-1} \left( d + \frac{B}{2\Psi} + \frac{C_2}{C_1 + C_2 \varphi} \right)^{-1} \\ &\quad + \alpha_{-2} \left( d + \frac{B}{2\Psi} + \frac{C_2}{C_1 + C_2 \varphi} \right)^{-2}, \\ v_{26}(x, t) &= \alpha_0 + \alpha_{-1} \left( d + \frac{\sqrt{\Delta}}{\Psi} \coth \left( \frac{\sqrt{\Delta}}{\Psi} \varphi \right) \right)^{-1} \\ &\quad + \alpha_{-2} \left( d + \frac{\sqrt{\Delta}}{\Psi} \coth \left( \frac{\sqrt{\Delta}}{\Psi} \varphi \right) \right)^{-2}, \\ v_{27}(x, t) &= \alpha_0 + \alpha_{-1} \left( d + \frac{\sqrt{\Delta}}{\Psi} \tanh \left( \frac{\sqrt{\Delta}}{\Psi} \varphi \right) \right)^{-1} \\ &\quad + \alpha_{-2} \left( d + \frac{\sqrt{\Delta}}{\Psi} \tanh \left( \frac{\sqrt{\Delta}}{\Psi} \varphi \right) \right)^{-2}, \end{aligned}$$



$$v_{28}(x, t) = \alpha_0 + \alpha_{-1} \left( d + \frac{\sqrt{-\Delta}}{\Psi} \cot \left( \frac{\sqrt{-\Delta}}{\Psi} \varphi \right) \right)^{-1} \\ + \alpha_{-2} \left( d + \frac{\sqrt{-\Delta}}{\Psi} \cot \left( \frac{\sqrt{-\Delta}}{\Psi} \varphi \right) \right)^{-2}, \\ v_{29}(x, t) = \alpha_0 + \alpha_{-1} \left( d - \frac{\sqrt{-\Delta}}{\Psi} \tan \left( \frac{\sqrt{-\Delta}}{\Psi} \varphi \right) \right)^{-1} \\ + \alpha_{-2} \left( d - \frac{\sqrt{-\Delta}}{\Psi} \tan \left( \frac{\sqrt{-\Delta}}{\Psi} \varphi \right) \right)^{-2},$$

where  $\varphi = x + Wt$ .

For Case 3: similarly, substituting Eq. (18) into Eq. (15), together with Eqs. (7)–(11) and simplifying, our obtained travelling wave solutions become (if  $C_1 = 0$  but  $C_2 \neq 0$ ;  $C_2 = 0$  but  $C_1 \neq 0$ , for 1<sup>st</sup> two solutions, again these conditions for  $v_{33}$  and  $v_{34}$ , also same conditions could be applied for solutions  $v_{36}$  and  $v_{37}$ , moreover, mentioned conditions are implemented to solutions  $v_{38}$  and  $v_{39}$ ) respectively:

$$v_{31}(x, t) = \alpha_0 + \frac{\alpha_2 \Omega}{4\Psi^2} \coth^2 \left( \frac{\sqrt{\Omega}}{2\Psi} \varphi \right) + \frac{4\Psi^2 \Phi}{\Omega} \tanh^2 \left( \frac{\sqrt{\Omega}}{2\Psi} \varphi \right),$$

where  $\alpha_0 = \frac{2bW(4\Delta+B^2)+A^2(1+W)}{2aA^2}$ ,  $\Phi = \frac{-3bW(8\Delta(2\Delta+B^2)+B^4)}{8aA^2\Psi^2}$  and  $\varphi = x + Wt$ .

$$v_{32}(x, t) = \alpha_0 + \frac{\alpha_2 \Omega}{4\Psi^2} \tanh^2 \left( \frac{\sqrt{\Omega}}{2\Psi} \varphi \right) + \frac{4\Psi^2 \Phi}{\Omega} \coth^2 \left( \frac{\sqrt{\Omega}}{2\Psi} \varphi \right), \\ v_{33}(x, t) = \alpha_0 - \frac{\Omega \alpha_2}{4\Psi^2} \cot^2 \left( \frac{\sqrt{-\Omega}}{2\Psi} \varphi \right) - \frac{4\Psi^2 \Phi}{\Omega} \tan^2 \left( \frac{\sqrt{-\Omega}}{2\Psi} \varphi \right), \\ v_{34}(x, t) = \alpha_0 - \frac{\Omega \alpha_2}{4\Psi^2} \tan^2 \left( \frac{\sqrt{-\Omega}}{2\Psi} \varphi \right) - \frac{4\Psi^2 \Phi}{\Omega} \cot^2 \left( \frac{\sqrt{-\Omega}}{2\Psi} \varphi \right), \\ v_{35}(x, t) = \alpha_0 + \alpha_2 \left( \frac{C_2}{C_1 + C_2 \varphi} \right)^2 + \Phi \left( \frac{C_2}{C_1 + C_2 \varphi} \right)^{-2} \\ v_{36}(x, t) = \alpha_0 + \alpha_2 \left( \frac{\sqrt{\Delta}}{\Psi} \coth \left( \frac{\sqrt{\Delta}}{\Psi} \varphi \right) - \frac{B}{2\Psi} \right)^2 \\ + \Phi \left( \frac{\sqrt{\Delta}}{\Psi} \coth \left( \frac{\sqrt{\Delta}}{\Psi} \varphi \right) - \frac{B}{2\Psi} \right)^{-2}, \\ v_{37}(x, t) = \alpha_0 + \alpha_2 \left( \frac{\sqrt{\Delta}}{\Psi} \tanh \left( \frac{\sqrt{\Delta}}{\Psi} \varphi \right) - \frac{B}{2\Psi} \right)^2 \\ + \Phi \left( \frac{\sqrt{\Delta}}{\Psi} \tanh \left( \frac{\sqrt{\Delta}}{\Psi} \varphi \right) - \frac{B}{2\Psi} \right)^{-2}, \\ v_{38}(x, t) = \alpha_0 + \alpha_2 \left( \frac{\sqrt{-\Delta}}{\Psi} \coth \left( \frac{\sqrt{-\Delta}}{\Psi} \varphi \right) - \frac{B}{2\Psi} \right)^2 \\ + \Phi \left( \frac{\sqrt{-\Delta}}{\Psi} \coth \left( \frac{\sqrt{-\Delta}}{\Psi} \varphi \right) - \frac{B}{2\Psi} \right)^{-2}, \\ v_{39}(x, t) = \alpha_0 + \alpha_2 \left( \frac{\sqrt{-\Delta}}{\Psi} \tanh \left( \frac{\sqrt{-\Delta}}{\Psi} \varphi \right) + \frac{B}{2\Psi} \right)^2 \\ + \Phi \left( \frac{\sqrt{-\Delta}}{\Psi} \tanh \left( \frac{\sqrt{-\Delta}}{\Psi} \varphi \right) + \frac{B}{2\Psi} \right)^{-2}.$$

where  $\varphi = x + Wt$ .

For case 4: similarly, substituting Eq. (19) into Eq. (15), along with Eqs. (7)–(11) and simplifying, the wave solutions become (if  $C_1 = 0$  but  $C_2 \neq 0$ ;  $C_2 = 0$  but  $C_1 \neq 0$ , for 1<sup>st</sup> two

solutions, same conditions for  $v_{43}$  and  $v_{44}$ , also these conditions for solutions  $v_{46}$  and  $v_{47}$ , in addition, mentioned conditions are executed to solutions  $v_{48}$  and  $v_{49}$ ) respectively:

$$v_{41}(x, t) = \alpha_0 + \frac{4\alpha_{-2}\Psi^2}{\Omega} \tanh^2 \left( \frac{\sqrt{\Omega}}{2\Psi} \varphi \right), \\ v_{42}(x, t) = \alpha_0 + \frac{4\alpha_{-2}\Psi^2}{\Omega} \coth^2 \left( \frac{\sqrt{\Omega}}{2\Psi} \varphi \right), \\ v_{43}(x, t) = \alpha_0 - \frac{4\alpha_{-2}\Psi^2}{\Omega} \tan^2 \left( \frac{\sqrt{\Omega}}{2\Psi} \varphi \right), \\ v_{44}(x, t) = \alpha_0 - \frac{4\alpha_{-2}\Psi^2}{\Omega} \cot^2 \left( \frac{\sqrt{\Omega}}{2\Psi} \varphi \right), \\ v_{45}(x, t) = \alpha_0 + \alpha_{-2} \left( \frac{C_2}{C_1 + C_2 \varphi} \right)^{-2}, \\ v_{46}(x, t) = \alpha_0 + \alpha_{-2} \left( \frac{-B}{2\Psi} + \frac{\sqrt{\Delta}}{\Psi} \coth \left( \frac{\sqrt{\Delta}}{\Psi} \varphi \right) \right)^{-2}, \\ v_{47}(x, t) = \alpha_0 + \alpha_{-2} \left( \frac{-B}{2\Psi} + \frac{\sqrt{\Delta}}{\Psi} \tanh \left( \frac{\sqrt{\Delta}}{\Psi} \varphi \right) \right)^{-2}, \\ v_{48}(x, t) = \alpha_0 + \alpha_{-2} \left( \frac{-B}{2\Psi} + \frac{\sqrt{-\Delta}}{\Psi} \cot \left( \frac{\sqrt{-\Delta}}{\Psi} \varphi \right) \right)^{-2}, \\ v_{49}(x, t) = \alpha_0 + \alpha_{-2} \left( \frac{-B}{2\Psi} + \frac{\sqrt{-\Delta}}{\Psi} \tan \left( \frac{\sqrt{-\Delta}}{\Psi} \varphi \right) \right)^{-2},$$

where  $\varphi = x + Wt$ .

#### 4. Discussions

The advantages and reliability of the proposed method over the basic ( $G'/G$ )-expansion method, the improved ( $G'/G$ )-expansion method, the generalized and improved ( $G'/G$ )-expansion method, new approach of the ( $G'/G$ )-expansion method and new approach of the generalized ( $G'/G$ )-expansion method have been described in the following.

*Advantages:* It is important to point out that, the significant advantages of further extension of the generalized and improved ( $G'/G$ )-expansion method over the above mentioned methods are that, the proposed method provides more general and a rich structure of new travelling wave solutions including many parameters. Furthermore, if parameters replace by particular values, some of our solutions coincide with the published results, which validates our newly generated solutions and other solutions are not reported in the previous literature.

*Validity:* The presentation of the solutions compared with the form of the presentations of Wang et al. (2008), Zhang et al. (2010), Akbar et al. (2012) and Naher and Abdullah (2013)) is as follows:

- (i) if  $A = 1, B$  takes  $-\lambda, C = 0$  and  $E$  takes  $-\mu$ , in Eq. (6), the nonlinear ODE (6) coincides with linear ODE Eq. (2.5) of Wang et al. (2008), Eq. (4) of Zhang et al. (2010), Eq. (2.5) of Akbar et al. (2012),
- (ii) if  $\alpha_{-1} = 0, \alpha_{-1} = -\beta_1, \beta_1 = 0, \alpha_{-2} = 0, \alpha_{-2} = -\beta_2, \beta_2 = 0$  and  $d = 0$  in Eq. (15), then presentation of further extension of the generalized and improved ( $G'/G$ )-expansion method turns into the basic ( $G'/G$ )-expansion method, introduced by Wang et al. (2008),

- (iii) if  $\beta_1 = 0$ ,  $\beta_2 = 0$  and  $d = 0$  in Eq. (15), our solution form matched with Eq. (9) (Zhang et al., 2010),
- (iv) if  $\beta_1 = 0$  and  $\beta_2 = 0$  in Eq. (15), the presented solution form come out the generalized and improved ( $G'/G$ )-expansion method which was proposed by Akbar et al. (2012),
- (v) if  $\alpha_{-1} = 0$ ,  $\alpha_{-2} = 0$  and  $d = 0$  in Eq. (15), the solution form of travelling wave solutions turns into new approach of ( $G'/G$ )-expansion method, presented by Naher and Abdullah (2013),
- (vi) if  $\alpha_{-1} = 0$  and  $\alpha_{-2} = 0$  in Eq. (15), new presentation of the solutions come out new approach of generalized ( $G'/G$ )-expansion method, which was firstly proposed by Naher and Abdullah (2013).

The obtained solutions also compared with the published results of Zhang et al. (2010) and Akbar et al. (2012) are as follows:

- (i) if  $A = 1$ ,  $B = -\lambda$ ,  $C = 0$ ,  $E = -\mu$ ,  $\beta_1 = 0$ ,  $\beta_2 = 0$ ,  $d = 0$ ,  $W = V$  and  $K = C$  in Eq. (17), the present case 2 is identical with the case 1 [Eq. (10)] (Zhang et al., 2010),
- (ii) if  $A = 1$ ,  $B = -\lambda$ ,  $C = 0$ ,  $E = -\mu$ ,  $\beta_1 = 0$ ,  $\beta_2 = 0$ ,  $d = 0$ ,  $W = V$  and  $K = C$  in Eq. (16), our obtained case 1 coincided with the case 2 [Eq. (11)] (Zhang et al., 2010),
- (iii) if  $A = 1$ ,  $B = -\lambda$ ,  $C = 0$ ,  $E = -\mu$ ,  $\beta_1 = 0$ ,  $\beta_2 = 0$ ,  $W = V$  and  $K = C$  in Eq. (17), our obtained case 2 coincided with the case 1 [Eq. (3.21)] (Akbar et al., 2012),
- (iv) if  $A = 1$ ,  $B = -\lambda$ ,  $C = 0$ ,  $E = -\mu$ ,  $\beta_1 = 0$ ,  $\beta_2 = 0$ ,  $W = V$  and  $K = C$  in Eq. (16), our obtained case 1 coincided with the case 2 [Eq. (3.21)] (Akbar et al., 2012),
- (v) if  $A = 1$ ,  $B = -\lambda$ ,  $C = 0$ ,  $E = -\mu$ ,  $\beta_1 = 0$ ,  $\beta_2 = 0$ ,  $W = V$  and  $K = C$  in Eq. (18), our obtained case 3 coincided with the case 3 [Eq. (3.23)] (Akbar et al., 2012).

By using cases 1 and 2, Zhang et al. (2010) obtained solutions  $u1_1 - u1_3$  and solutions  $u2_1 - u2_3$ . After considering particular values for parameters, our solutions  $v1_1 - v1_5$  and  $v2_1 - v2_5$  are identical with the solutions obtained by Zhang et al. (2010). On the other hand, if we do not follow above restrictions (i) and (ii), solutions  $v1_1 - v1_5$  and  $v2_1 - v2_5$  are dissimilar with Zhang et al. (2010). Moreover, solutions  $v1_6 - v1_9, v2_6 - v2_9, v3_1 - v3_9$  and  $v4_1 - v4_9$  have been generated in this work, which were not constructed by Zhang et al. (2010).

Also, Akbar et al. (2012) generated solutions  $v1_1 - v1_5$ ,  $v2_1 - v2_5$  and  $v3_1 - v3_5$  by using case 1 to case 3. Newly generated solutions  $v1_1 - v1_5, v2_1 - v2_5$  and  $v3_1 - v3_5$  are coincided with solutions of Akbar et al. (2012), if our parameters take specific values. Without following above conditions (iii)–(v), our solutions  $v1_1 - v1_5, v2_1 - v2_5$  and  $v3_1 - v3_5$  are not same (Akbar et al., 2012). In addition, we obtained solutions  $v1_6 - v1_9, v2_6 - v2_9, v3_6 - v3_9$  and  $v4_1 - v4_9$ , which had not been reported by Akbar et al. (2012).

Therefore, we may state that the basic ( $G'/G$ )-expansion method; the improved ( $G'/G$ )-expansion method; the generalized and improved ( $G'/G$ )-expansion method; new approach of ( $G'/G$ )-expansion method; and new approach of generalized ( $G'/G$ )-expansion method are a particular case of our proposed method. It is also noticed that, further extension of the generalized and improved ( $G'/G$ )-expansion method is more effective and trustworthy for generating abundant new travelling wave solutions.

## 5. Conclusions

In this article, further extension of the generalized and improved ( $G'/G$ )-expansion method has been applied effectively to the ZKBBM equation. New auxiliary equation is used involving many arbitrary parameters; additional parameter is also executed in the method. The parameters can take any real values and nonlinear ordinary differential equation can produce many solutions. Each solution has rich physical structures. Obtained solutions show that the proposed method is more effective, concise and straightforward than the earliest methods and can be applied for many nonlinear PDEs in mathematical physics and engineering sciences.

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