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المرونة الحرارية المغناطيسية ذات الرتبين الكسريتين للانتقال الحراري

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الملخص:

تم بناء نموذج جديد في نظرية المرونة الحرارية المغناطيسية لدرجات الحرارة في حالة قانون فوريير للانتشار الحراري الذي يعتمد على كل من التفاضلات والتكاملات الكسرية بالنسبة للزمن لوسط تام الموصلية الكهربائية. طبق هذا النموذج على مسألة أحادية البعد لنصف الفراغ تام الموصلية الكهربائية في وجود مجال مغناطيسي ثابت ومصادر حرارية. استخدمت طريقة فضاء الحالة وتحويلات لابلاس لإيجاد الحلول العامة لأي شروط حدية. وظفت الطرق العددية لإيجاد محولات لابلاس ومثلت النتائج بيانياً. عملت المقارنات للنتائج التي تم الحصول عليها لدراسة تأثير كل من معامل الرتبة الكسرية وتناقض درجة الحرارة على جميع المجالات محل الدراسة.



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ORIGINAL ARTICLE

Magneto-thermoelasticity with two fractional order heat transfer



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Abstract A new mathematical model of the equations of two-temperature magneto-thermoelasticity theory for a perfect conducting solid has been constructed in the context of a new consideration of heat conduction with a time-fractional derivative of order α ($0 < \alpha \leq 1$) and a time-fractional integral of order ν ($0 < \nu \leq 2$). This model is applied to one-dimensional problem for a perfect conducting half-space of elastic solid with heat source distribution in the presence of a constant magnetic field. Laplace transforms and state-space techniques will be used to obtain the general solution for any set of boundary conditions. A numerical method is employed for the inversion of the Laplace transforms. According to the numerical results and their graphs, conclusions about the new theory are given. Some comparisons are shown in figures to estimate the effects of the fractional order parameters and the temperature discrepancy on all the studied fields.

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1. Introduction

The classical uncoupled theory of thermoelasticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms; second, the heat equation is of a parabolic type, predicting infinite speeds of propagation for heat waves.

Biot (1956) introduced the theory of coupled thermoelasticity to overcome the first shortcoming. The governing equations for this theory are coupled, eliminating the first paradox of the classical theory. However, both theories share the second

shortcoming since the heat equation for the coupled theory is of a mixed parabolic–hyperbolic type.

The mathematical aspects of Lord–Shulman (1967) theory are explained and illustrated in detail in the work of Ignaczak and Ostoja-starzeweski (2009). Joseph and Preziosi (1990) state that the Cattaneo (1958) heat conduction law is the most obvious and simple generalization of the Fourier law that gives rise to a finite propagation speed.

Within the theoretical contributions to the subject are the proofs of uniqueness theorems under different conditions by Ignaczak (1979), Chandrasekharaiah (1984), Sherief (1987) and Ezzat and El-Karamany (2002). The fundamental solutions for generalized thermoelasticity problem were obtained Ezzat (2004).

The two-temperature thermoelasticity theory (2TT) and the classical theory of thermoelasticity (CTE) suffer from the

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so-called “paradox of heat conduction,” i.e., the heat equations for both theories of a mixed parabolic-hyperbolic type, predicting infinite speeds of propagation for heat waves contrary to physical observations. The generalized thermoelasticity theories in which the heat conduction equation is hyperbolic do not suffer from this paradox.

Chen and Gurtin (1968), Chen et al. (1968, 1969) have formulated a theory of heat conduction in deformable bodies, which depends on two distinct temperatures, the conductive temperature φ and thermodynamic temperature T . Ieşan (1970) established the uniqueness, reciprocity theorems and variational principle for homogeneous isotropic solid in the frame of coupled thermoelasticity theory involving two temperatures. Youssef (2006) extended this theory in the context of the generalized theory of thermoelasticity with one relaxation time and Magaña and Quintanilla (2009) studied modifications of the non-classical models of thermoelasticity. They proved uniqueness results for the solutions of the systems of equations that model both theories for anisotropic material.

The foundation of magnetoelasticity was presented by Kaliski and Petykiewicz (1959). Increasing attention is being devoted to the interaction between magnetic field and strain field in a thermoelastic solid due to its many applications in the fields of geophysics, plasma physics and related topics. In the preceding references, it was assumed that the interactions between the two fields take place by means of the Lorentz forces appearing in the equations of motion and by means of a term entering Ohm’s law and describing the electric field produced by velocity of a material charge, moving in a magnetic field.

Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering. The most important advantage of using fractional differential equations in these and other applications is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic and it is one reason why fractional calculus has become more and more popular (Caputo, 1967; Mainardi, 1997 and Podlubny, 1999).

Although the tools of fractional calculus were available and applicable to various fields of study, the investigation of the theory of fractional differential equations started quite recently (Caputo, 1967). The differential equations involving Riemann–Liouville differential operators of fractional order $0 < \alpha < 1$, appear to be important in modeling several physical phenomena (Kiryakova, 1994) and therefore seem to deserve an independent study of their theory parallel to the well-known theory of ordinary differential equations. Khalid et al. (2012) applied the homotopy perturbation method and variational iteration method to obtain the approximate solution of the harmonic wave propagation in a nonlinear magneto-thermoelasticity under the influence of rotation. The analytical approximate solution for non-linear space-time fractional Klein–Gordon equation is given by Khalid and Mohamed (2013). A domain decomposition method to obtain approximate solutions for fractional PDEs was given by Khalid (2012).

Fractional calculus has been used successfully to modify many existing models of physical processes. The first

application of fractional derivatives was given by Abel who applied fractional calculus in the solution of an integral equation that arises in the formulation of the Tautochrone problem. The generalization of the concept of derivative and integral to a non-integer order has been subjected to several approaches and some various alternative definitions of fractional derivatives appeared in Refs. (Miller and Ross, 1993; Gorenflo and Mainardi, 1997, Hilfer, 2000). In the last few years fractional calculus was applied successfully in various areas to modify many existing models of physical processes, e.g., chemistry, biology, modeling and identification, electronics, wave propagation and viscoelasticity (Caputo and Mainardi, 1971; Caputo, 1974; and Rossikhin and Shitikova, 1997). One can refer to Podlubny (1999) for a survey of applications of fractional calculus.

Sherief et al. (2010) introduced a formula of heat conduction as

$$\mathbf{q} + \tau_o \frac{\partial^\alpha \mathbf{q}}{\partial t^\alpha} = -\kappa \nabla T, \quad 0 < \alpha \leq 1, \quad (1)$$

and proved a uniqueness theorem and derived a reciprocity relation and a variational principle.

Youssef (2010) introduced another formula of heat conduction in the form

$$\mathbf{q} + \tau_o \frac{\partial \mathbf{q}}{\partial t} = -k \Gamma^{v-1} \nabla T, \quad 0 < v \leq 2, \quad (2)$$

and a uniqueness theorem has been proved.

Ezzat (2011) established a new model of fractional heat conduction equation by using the new Taylor series expansion of time-fractional order which was developed by Jumarie (2010) as

$$\mathbf{q} + \frac{\tau_o^\alpha}{\alpha!} \frac{\partial^\alpha \mathbf{q}}{\partial t^\alpha} = -k \nabla T, \quad 0 < \alpha \leq 1. \quad (3)$$

Fractional order theory of a perfect conducting thermoelastic medium was investigated by Ezzat and El-Karamany (2011). El-Karamany and Ezzat (2011) introduced two models where the fractional derivatives and integrals are used to modify the Cattaneo heat-conduction law and, in the context of the two-temperature thermoelasticity theory, uniqueness and reciprocal theorems are proved, the convolutional variational principle is given and used to prove a uniqueness theorem with no restrictions imposed on the elasticity or thermal conductivity tensors, except symmetry conditions. The integral analog of the Leibniz rule for fractional calculus and its applications is derived by Jaimini et al. (2001).

In the current work, a new model of time fractional derivative of order α and time fractional integral of order v in heat conduction equation has been derived in the context of generalized thermoelasticity theory. The resulting formulation is applied to a semi-infinite electrically perfect conducting half-space of elastic solid in the presence of a constant magnetic field. The Laplace transform technique is used throughout. Laplace transforms are obtained using the complex inversion formula of the transform together with Fourier expansion techniques proposed by Honig and Hirdes (1984). The effects of various physical parameters on various heat transfer, stress, displacement and strain characteristics as well as the electric field are discussed in detail and represented graphically.

2. Two-temperature theory of thermoelasticity with two fractional order heat transfer

The conventional thermoelasticity is based on the principles of the classical theory of heat conductivity, specifically on the modified Fourier's law, in which relates the heat flux vector \mathbf{q} to the temperature gradient

$$\mathbf{q} = -kI^{v-1}\nabla T, \quad 0 < v \leq 2. \quad (4)$$

The energy equation in terms of the heat flux vector \mathbf{q} , (Povstenko, 2004)

$$\frac{\partial}{\partial t}(\rho C_E \theta + \gamma T_0 e) = -\nabla \cdot \mathbf{q} + Q. \quad (5)$$

During the past three decades, nonclassical thermoelasticity theories, in which Fourier law (4) and heat Eq. (5) are replaced by more general equations, have been formulated by taking Taylor's series to expand $\mathbf{q}(\mathbf{x}, t + \tau_o)$ and retaining terms up to the first order in τ_o . The first well-known generalization of such a type

$$\mathbf{q} + \tau_o \frac{\partial \mathbf{q}}{\partial t} = -kI^{v-1}\nabla T, \quad 0 < v \leq 2, \quad (6)$$

leads to the heat transport equation in the theory of generalized thermoelasticity (Youssef, 2010)

$$\begin{aligned} \frac{\partial}{\partial t} \left(1 + \tau_o \frac{\partial}{\partial t} \right) (\rho C_E T + \gamma T_0 e) \\ = kI^{v-1}\nabla^2 T + Q + \tau_o \frac{\partial Q}{\partial t}, \quad 0 < v \leq 2. \end{aligned} \quad (7)$$

Recently, Youssef (2006) investigated two-temperature generalized thermoelasticity theory in which Fourier law (4) is replaced by

$$\mathbf{q} + \tau_o \frac{\partial \mathbf{q}}{\partial t} = -kI^{v-1}\nabla \varphi, \quad 0 < v \leq 2, \quad (8)$$

$$\phi - T = \varphi - \Theta = a\nabla^2 \varphi, \quad (9)$$

leads to the heat equation in the form

$$\begin{aligned} \frac{\partial}{\partial t} \left(1 + \tau_o \frac{\partial}{\partial t} \right) (\rho C_E \Theta + \gamma \varphi_0 e) \\ = kI^{v-1}\nabla^2 \varphi + Q + \tau_o \frac{\partial Q}{\partial t}, \quad 0 < v \leq 2. \end{aligned} \quad (10)$$

In the current work, the new fractional Taylor's series of time-fractional order α developed in (Jumarie, 2010) is adopted to expand $\mathbf{q}(\mathbf{x}, t + \tau_o)$ and retaining terms up to order α in the thermal relaxation time τ_o , we get (Ezzat, 2011)

$$\mathbf{q}(\mathbf{x}, t + \tau_o) = \mathbf{q}(\mathbf{x}, t) + \frac{\tau_o^\alpha}{\alpha!} \frac{\partial^\alpha \mathbf{q}}{\partial t^\alpha}, \quad 0 < \alpha \leq 1. \quad (11)$$

From a mathematical viewpoint, Fourier law (11) in the theory of generalized fractional heat conduction involving two temperatures, is given by

$$\mathbf{q} + \frac{\tau_o^\alpha}{\alpha!} \frac{\partial^\alpha \mathbf{q}}{\partial t^\alpha} = -kI^{v-1}\nabla \varphi, \quad 0 < \alpha \leq 1, \quad 0 < v \leq 2. \quad (12)$$

Taking the partial time- derivative of fraction order α of Eq. (5), we get

$$\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} (\rho C_E \Theta + \gamma \varphi_0 e) = -\nabla \cdot \left(\frac{\partial^\alpha \mathbf{q}}{\partial t^\alpha} \right) + \frac{\partial^\alpha Q}{\partial t^\alpha}, \quad 0 < \alpha \leq 1. \quad (13)$$

Multiplying Eq. (13) by $\frac{t^\alpha}{\alpha!}$ and adding to Eq. (5) we have

$$\begin{aligned} \frac{\partial}{\partial t} \left(1 + \frac{\tau_o^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) (\rho C_E \Theta + \gamma \varphi_0 e) \\ = -\nabla \cdot \left(\mathbf{q} + \frac{\tau_o^\alpha}{\alpha!} \frac{\partial^\alpha \mathbf{q}}{\partial t^\alpha} \right) + Q + \frac{\tau_o^\alpha}{\alpha!} \frac{\partial^\alpha Q}{\partial t^\alpha}, \quad 0 < \alpha \leq 1. \end{aligned} \quad (14)$$

Substituting from Eq. (14), we get

$$\begin{aligned} \frac{\partial}{\partial t} \left(1 + \frac{\tau_o^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) (\rho C_E \Theta + \gamma \varphi_0 e) \\ = kI^{v-1}\nabla^2 \varphi + \left(1 + \frac{\tau_o^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) Q, \quad 0 < \alpha \leq 1, \quad 0 < v \leq 2. \end{aligned} \quad (15)$$

Eq. (15) is the generalized energy equation with fractional time derivatives and integrals in which the relaxation time τ_o is considered. Some theories of heat conduction law follow as limit cases for different values of the parameters α , v and τ_o .

Limiting cases

(i) In the theory of thermoelasticity

1- The heat Eq. (21) in the limiting case $\Theta = \varphi$, $\tau_o = 0$ and $v = 1$ transforms to the work of Biot (1956)..

2- The heat Eq. (21) in the limiting case $\Theta = \varphi$, $\tau_o = 0$ and $0 < v \leq 2$ transforms to the work of Povstenko (2004)

(ii) In the theory of two temperature thermoelasticity

3- The heat Eq. (20) in the limiting case $\Theta \neq \varphi$, $\tau_o = 0$ and $v = 1$ transforms to the work of Chen and Gurtin (1968) and Ieşan (1970)..

(iii) In the theory of generalized thermoelasticity

4- The heat Eq. (21) in the limiting case $\Theta = \varphi$, $v = 1$, $\tau_o > 0$ and $\alpha = 1$ transforms to the work of Lord and Shulman (1967).

(iv) In the theory of two temperature generalized thermoelasticity

5- The heat Eq. (21) in the limiting case $\Theta \neq \varphi$, $v = 1$, $\tau_o > 0$ and $\alpha = 1$ transforms to the works of Youssef (2006).

6- (v) In the theory of two temperature generalized magneto-thermoelasticity

7- The heat Eq. (21) in the limiting case $\Theta \neq \varphi$, $v = 1$, $\tau_o > 0$ and $\alpha = 1$ transforms to the works of Ezzat et al. (2009).

8- (vi) In the theory of generalized thermoelasticity with derivative fractional order α

9- The heat Eq. (21) in the limiting case $\Theta = \varphi$, $v = 1$, $\tau_o > 0$ and $0 < \alpha \leq 1$ transforms to the work of Sherief et al. (2010) and Ezzat (2011).

(vii) In the theory of generalized thermoelasticity with integral fractional order v

10- The heat Eq. (21) in the limiting case, $\Theta = \varphi$, $\tau_o > 0$, $\alpha = 1$ and $0 < v \leq 2$ transforms to the work of Youssef (2010).

3. The physical problem and state space approach

We shall consider a thermoelastic medium of perfect conductivity permeated by an initial magnetic field \mathbf{H} . This produces an induced magnetic field \mathbf{h} and induced electric field \mathbf{E} , which satisfy the linearized equations of electromagnetism and are valid for slowly moving media. The governing equations for generalized two-temperature magneto-thermoelasticity consist of (Ezzat, 2001)

$$\text{curl } \mathbf{h} = \mathbf{J} + \varepsilon_o \frac{\partial \mathbf{E}}{\partial t}, \quad (16)$$

$$\text{curl } \mathbf{E} = -\mu_o \frac{\partial \mathbf{h}}{\partial t}, \quad (17)$$

$$\mathbf{E} = -\mu_o \left(\frac{\partial \mathbf{u}}{\partial t} \wedge \mathbf{H} \right), \quad (18)$$

$$\text{div } \mathbf{h} = 0. \quad (19)$$

The equation of motion in the absence of body forces

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sigma_{ji,j} + \mu_o (J \wedge H)_i, \quad (20)$$

The heat equation with fractional time derivatives

$$\left(1 + \frac{\tau_o^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) \left(\rho C_E \frac{\partial \theta}{\partial t} \Theta + \varphi_o \gamma \frac{\partial e}{\partial t} - Q \right) = k I^{n-1} \varphi_{,ii}, \quad 0 < \alpha \leq 1, \quad 0 < v \leq 2. \quad (21)$$

The relation between the heat conduction and dynamical heat

$$\varphi - \Theta = a \varphi_{,ii} \quad (22)$$

where $a > 0$, is the temperature discrepancy. The constitutive equation

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e \delta_{ij} - \gamma \Theta \delta_{ij}. \quad (23)$$

The strain-displacement relations

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}). \quad (24)$$

In the above equations a comma denotes material derivatives and the summation convention is used.

Now, we shall consider a homogeneous isotropic thermoelastic perfect conducting solid occupying half-space $x \geq 0$, which is initially quiescent and where all the state functions depend only on the dimension x and the time t and the displacement vector has components $(u(x, t), 0, 0)$. A constant magnetic field with components $(0, H_o, 0)$ is permeating the medium. The induced magnetic field \mathbf{h} will have one component in the y -direction, while the induced electric field \mathbf{E} will have one component in z -direction.

Let us introduce the following non-dimensional variables:

$$\begin{aligned} x' &= c_o \eta_o x, u' = c_o \eta_o u, t' = c_o^2 \eta_o t, \tau'_o = c_o^2 \eta_o \tau_o, \Theta' = \frac{\gamma \Theta}{\rho c_o^2}, \\ \sigma'_{ij} &= \frac{\sigma_{ij}}{\rho c_o^2}, q'_i = \frac{q_i \gamma}{k \rho c_o^3 \eta_o}, q'_i = \frac{q_i \gamma}{k \rho c_o^3 \eta_o}, h' = \frac{h}{H_o}, E' = \frac{E}{\mu_o H_o c_o}, \\ \varphi' &= \frac{\gamma \varphi}{\rho c_o^2}, \eta_o = \frac{\rho C_E}{\kappa}, \varepsilon = \frac{\varphi_o \gamma^2}{k \rho c_o^2 \eta_o}, Q' = \frac{Q \gamma}{k \rho c_o^4 \eta_o^2} \end{aligned}$$

The dimensionless temperature discrepancy is $\beta_o = a c_o^2 \eta_o^2 = a (c_o/k)^2$.

Using homogeneity and scale change of fractional derivatives, the following system of equations in terms of the preceding non-dimensional variables results (suppressing the primes for convenience)

$$h = -e, \quad (25)$$

$$E = -\frac{\partial u}{\partial t}, \quad (26)$$

$$I^{n-1} \frac{\partial^2 \varphi}{\partial x^2} = \left(\frac{\partial}{\partial t} + \frac{\tau_o^\alpha}{\alpha!} \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} \right) (\Theta + \varepsilon e) - \left(1 + \frac{\tau_o^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) Q, \quad (27)$$

$$\frac{\partial^2 \sigma}{\partial x^2} + \beta \frac{\partial^2 e}{\partial x^2} = \alpha^* \frac{\partial^2 e}{\partial t^2}, \quad (28)$$

$$\sigma = e - \Theta, \quad (29)$$

$$\varphi - \Theta = \beta_o \frac{\partial^2 \varphi}{\partial x^2}, \quad (30)$$

where $\alpha^* = 1 + \frac{\varepsilon^2}{\beta}$ and $\beta = (\alpha_o/c_o)^2$.

From now on, we shall consider a heat source of the form $Q = Q_o \delta(x) H(t)$, where Q_o is a positive constant.

To simplify the algebra, only problems with zero initial conditions are considered. Applying the Laplace transform defined by the formulas (Povstenko, 2005)

$$\left. \begin{aligned} L\{g(t)\} &= \bar{g}(s) = \int_0^\infty e^{-st} g(t) dt \\ L\{D^n g(t)\} &= s^n L\{g(t)\} \quad n > 0 \end{aligned} \right\}, \quad (31)$$

on both sides of Eqs. (25)–(30) and writing the resulting equations in matrix form results in

$$\frac{d^2}{dx^2} \begin{Bmatrix} \bar{\varphi} \\ \bar{\sigma} \end{Bmatrix} = \begin{Bmatrix} L_1 & L_2 \\ M_1 & M_2 \end{Bmatrix} \begin{Bmatrix} \bar{\varphi} \\ \bar{\sigma} \end{Bmatrix} - Q_o \beta^* \delta(x) \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad (32)$$

where

$$\begin{aligned} \varpi &= s^\nu \left(1 + \frac{\tau_o^\alpha}{\alpha!} s^\alpha \right), \quad \beta^* = \frac{\varpi}{s[1 + \beta_o \varpi(1 + \varepsilon)]}, \\ L_1 &= \frac{\varpi(1 + \varepsilon)}{1 + \beta_o \varpi(1 + \varepsilon)}, \quad L_2 = \frac{\varpi \varepsilon}{1 + \beta_o \varpi(1 + \varepsilon)} \\ M_1 &= \frac{m(\alpha^* s^2 - \beta L_1)}{1 + \beta n}, \quad M_2 = \frac{n(\alpha^* s^2 - \beta L_2)}{1 + \beta n}, \\ m &= 1 - \beta_o L_1 \quad \text{and} \quad n = 1 - \beta_o L_2. \end{aligned}$$

Choosing as state variables the temperature of heat conduction $\bar{\varphi}$ and the stress component $\bar{\sigma}$ in the x -direction, Eq. (32) can be written in the absence of heat source as:

$$\frac{d^2 \bar{G}(x, s)}{dx^2} = A(s) \bar{G}(x, s), \quad (33)$$

where

$$\bar{G}(x, s) = \begin{bmatrix} \bar{\varphi}(x, s) \\ \bar{\sigma}(x, s) \end{bmatrix} \quad \text{and} \quad A(s) = \begin{bmatrix} L_1 & L_2 \\ M_1 & M_2 \end{bmatrix}.$$

The formal solution of system (33) can be written in the form

$$\bar{G}(x, s) = \exp \left[-\sqrt{A(s)} x \right] \bar{G}(0, s), \quad (34)$$

where

$$\bar{G}(0, s) = \begin{bmatrix} \bar{\varphi}(0, s) \\ \bar{\sigma}(0, s) \end{bmatrix} = \begin{bmatrix} \bar{\varphi}_o \\ \bar{\sigma}_o \end{bmatrix}$$

where for bounded solution with large x , we have canceled the part of exponential that has a positive power.

We shall use the well-known Cayley–Hamilton theorem to find the form of the matrix $\exp[-\sqrt{A(s)}x]$. The characteristic equation of the matrix $A(s)$ can be written as follows:

$$k^2 - k(L_1 + M_2) + (L_1M_2 - L_2M_1) = 0. \quad (35)$$

The roots of this equation, namely, k_1 and k_2 , satisfy the following relations:

$$k_1 + k_2 = L_1 + M_2 \quad (36a)$$

$$k_1k_2 = L_1M_2 - L_2M_1. \quad (36b)$$

The Taylor series expansion of the matrix exponential in Eq. (34) has the form

$$\exp[-\sqrt{A(s)}x] = \sum_{n=0}^{\infty} \frac{[-\sqrt{A(s)}x]^n}{n!}. \quad (37)$$

$$\bar{\Theta}(x, s) = \frac{(k_1\bar{\varphi}_o - L_1\bar{\varphi}_o - L_2\bar{\sigma}_o)(1 - \beta_o k_2)e^{-\sqrt{k_2}x} - (k_2\bar{\varphi}_o - L_1\bar{\varphi}_o - L_2\bar{\sigma}_o)(1 - \beta_o k_1)e^{-\sqrt{k_1}x}}{k_1 - k_2} \quad (44)$$

Using the Cayley–Hamilton theorem, we can express A^2 and higher powers of the matrix A in terms of I and A , where I is the unit matrix of second order.

Thus, the infinite series in Eq. (37) can be reduced to

$$\exp[-\sqrt{A(s)}x] = a_o(x, s)I + a_1(x, s)A(s). \quad (38)$$

where a_o and a_1 are coefficients depending on x and s .

By the Cayley–Hamilton theorem, the characteristic roots k_1 and k_2 of the matrix A must satisfy Eq. (37), thus

$$\exp[-\sqrt{k_1}x] = a_o + a_1k_1, \quad (39a)$$

$$\exp[-\sqrt{k_2}x] = a_o + a_1k_2. \quad (39b)$$

The solution of the above system is given by

$$a_o = \frac{k_1e^{-\sqrt{k_2}x} - k_2e^{-\sqrt{k_1}x}}{k_1 - k_2}, \quad \text{and} \quad a_1 = \frac{e^{-\sqrt{k_1}x} - e^{-\sqrt{k_2}x}}{k_1 - k_2}.$$

Hence, we have

$$\exp[-\sqrt{k_1}x] = L_{ij}(x, s), \quad i, j = 1, 2,$$

where

$$L_{11} = \frac{e^{-\sqrt{k_2}x}(k_1 - L_1) - e^{-\sqrt{k_1}x}(k_2 - L_1)}{k_1 - k_2},$$

$$L_{12} = \frac{L_2(e^{-\sqrt{k_1}x} - e^{-\sqrt{k_2}x})}{k_1 - k_2}$$

$$L_{22} = \frac{e^{-\sqrt{k_1}x}(k_2 - M_2) - e^{-\sqrt{k_2}x}(k_1 - M_2)}{k_1 - k_2},$$

$$L_{21} = \frac{M_1(e^{-\sqrt{k_1}x} - e^{-\sqrt{k_2}x})}{k_1 - k_2}. \quad (40)$$

The solution in Eq. (34) can be written in the form

$$\bar{G}(x, s) = L_{ij}\bar{G}(0, s). \quad (41)$$

Hence, we obtain

$$\bar{\varphi}(x, s) = \frac{(k_1\bar{\varphi}_o - L_1\bar{\varphi}_o - L_2\bar{\sigma}_o)e^{-\sqrt{k_2}x} - (k_2\bar{\varphi}_o - L_1\bar{\varphi}_o - L_2\bar{\sigma}_o)e^{-\sqrt{k_1}x}}{k_1 - k_2}, \quad (42)$$

$$\bar{\sigma}(x, s) = \frac{(k_1\bar{\sigma}_o - M_1\bar{\varphi}_o - M_2\bar{\sigma}_o)e^{-\sqrt{k_2}x} - (k_2\bar{\sigma}_o - M_1\bar{\varphi}_o - M_2\bar{\sigma}_o)e^{-\sqrt{k_1}x}}{k_1 - k_2}. \quad (43)$$

By using Eqs. (42) and (43) with Eq. (29) we get

4. Application

It should be noted that the corresponding expressions for two-temperature generalized thermoelasticity with relaxation time in the absence of magnetic field can be deduced by setting $\alpha_o = 0$ in Eq. (40).

We consider a semi-space homogeneous medium of perfect conductivity occupying the region $x \geq 0$ with quiescent initial state and boundary conditions in the following form:

- (i) Thermal boundary condition: A thermal shock is applied to the boundary plane $x = 0$ in the form

$$\varphi(0, t) = \varphi_o H(t), \quad \text{or} \quad \bar{\varphi}(0, s) = \bar{\varphi}_o = \frac{\varphi_o}{s}, \quad (45)$$

where $\bar{\varphi}_o$ is a constant and $H(t)$ is the Heaviside unit step function.

- (i) Mechanical boundary condition: The bounding plane $x = 0$ is taken to be traction-free, i.e.

$$\sigma(0, t) + T_{11}(0, t) - T_{11}^o(0, t) = 0, \quad (46)$$

where T_{11}^o is the Maxwell stress tensor in a vacuum.

Since the transverse components of the vectors E and h are continuous across the bounding plane, i.e. $E(0, t) = E^o(0, t)$ and $h(0, t) = h^o(0, t)$, $t > 0$, where E^o and h^o are the components of the induced electric and magnetic field in free space and the relative permeability is very nearly unity, it follows that $T_{11}(0, t) = T_{11}^o(0, t)$ and Eq. (46) reduces to (Ezzat, 2011)

$$\sigma(0, t) = 0, \quad \text{or} \quad \bar{\sigma}(0, s) = \bar{\sigma}_o = 0. \quad (47)$$

Hence, we can use the conditions on (45) and (46) into Eqs. (42) and (43) to get the exact solution in the Laplace transform domain in the following forms:

$$\bar{\varphi}(x,s) = \frac{\varphi_o \left[(k_1 - L_1)e^{-\sqrt{k_2}x} - (k_2 - L_1)e^{-\sqrt{k_1}x} \right]}{s(k_1 - k_2)}, \quad (48)$$

$$\bar{\sigma}(x,s) = \frac{\varphi_o M_1 \left(e^{-\sqrt{k_1}x} - e^{-\sqrt{k_2}x} \right)}{s(k_1 - k_2)}, \quad (49)$$

$$\bar{\Theta}(x,s) = \frac{\varphi_o \left[B e^{-\sqrt{k_2}x} - A e^{-\sqrt{k_1}x} \right]}{s(k_1 - k_2)}, \quad (50)$$

$$\begin{aligned} \bar{e}(x,s) &= \frac{\varphi_o \left[[m(k_1 - L_1) - nM_1] e^{-\sqrt{k_2}x} - [m(k_2 - L_1) - nM_1] e^{-\sqrt{k_1}x} \right]}{s(k_1 - k_2)}, \end{aligned} \quad (51)$$

where $A = (k_2 - L_1)(1 - \beta_o k_1)$, $B = (k_1 - L_1)(1 - \beta_o k_2)$.

From Eq. (24), the displacement takes the form:

$$\bar{u}(x,s) = \frac{\varphi_o \left(C e^{-\sqrt{k_1}x} - D e^{-\sqrt{k_2}x} \right)}{s(k_1 - k_2)}, \quad (52)$$

where $C = \frac{m(k_2 - L_1) - nM_1}{\sqrt{k_1}}$, $D = \frac{m(k_1 - L_1) - nM_1}{\sqrt{k_2}}$.

The induced magnetic and electric field takes the following forms

$$\begin{aligned} \bar{h}(x,s) &= \frac{\varphi_o \left[[m(k_1 - L_1) - nM_1] e^{-\sqrt{k_2}x} - [m(k_2 - L_1) - nM_1] e^{-\sqrt{k_1}x} \right]}{s(k_1 - k_2)}, \end{aligned} \quad (53)$$

$$\bar{E}(x,s) = -\frac{\varphi_o \left(C e^{-\sqrt{k_1}x} - D e^{-\sqrt{k_2}x} \right)}{s^2(k_1 - k_2)}. \quad (54)$$

Those complete the solution in the Laplace transform domain.

5. The numerical inversion of the Laplace transforms

In order to invert the Laplace transform in the above equations, we adopt a numerical inversion method based on a Fourier series expansion (Honig and Hirdes, 1984). In this method, the inverse $f(t)$ of the Laplace transform \bar{f} is approximated by the relation.

$$\begin{aligned} f(t) &= \frac{e^{ct}}{t_1} \left[\frac{1}{2} \bar{f}(c) + R1 \sum_{k=1}^N \bar{f} \left(c + \frac{ik\pi}{t_1} \right) \exp \left(\frac{ik\pi t}{t_1} \right) \right], \quad 0 \\ &\leq t \leq 2t, \end{aligned} \quad (55)$$

where N is a sufficiently large integer representing the number of terms in the truncated infinite Fourier series, N must be chosen such that

$$f(t) = e^{ct} R1 \left[\bar{f} \left(c + \frac{ik\pi}{t_1} \right) \exp \left(\frac{iN\pi t}{t_1} \right) \right] \leq \varepsilon_1,$$

where ε_1 is a persecuted small positive number that corresponds to the degree of accuracy to be achieved. The parameter c is a positive free parameter that must be greater than the real parts of all singularities of $\bar{f}(s)$, the optimal choice of c was

obtained according to the criteria described in Honig and Hirdes (1984).

6. Numerical results and discussion

The copper material was chosen for purposes of numerical evaluations. The constants of the problem were taken as following (Ezzat, 2011):(see Table 1)

The investigation of the effect of the fractional orders α and ν on perfect conducting thermoelastic material in the presence of a magnetic field has been carried out in the preceding sections. The computations were performed for a value of time, namely $t = 0.1$. The numerical technique outlined above was used to obtain the conductive temperature, the thermodynamic temperature, the stress, the displacement and the strain distributions. The results are represented graphically at different values of derivative fractional order α and integral fractional order ν as shown in Figs. 1–4 as well as at different positions of x as shown in Figs. 5–10. In these figures we noticed the difference in all functions for the values of α ($0 < \alpha \leq 1$) and ν ($0 < \nu \leq 2$) where the case of $\alpha = 1$ (normal conductivity) indicates the old situation, and the case $0 < \alpha < 1$ (weak conductivity), indicates the new theory. For a normal conductivity $\alpha = 1$, the results coincide with all the previous results of applications that are taken in the context of the generalized thermoelasticity with one relaxation time in the various fields. We observe the following:

- The fractional orders α and ν have a significant effect on all fields. The important phenomenon observed in all computations is that the solution to any of the functions considered for the new theory vanishes identically outside the surface region and the response to the thermal and mechanical effects does not reach infinity instantaneously but remains in a bounded region of the space. This result is very important that the fractional orders theory may preserve the advantage of both theories of Biot (1956) and Lord and Shulman (1967).
- * The fields are continuous functions for different values of the fractional orders α ($0 < \alpha < 1$) and ν ($0 < \nu \leq 2$) and are smoother in the case $\beta_o = 0.075$.
- * The waves reach the steady state depending on the value of the fractional orders α and ν as well as temperature discrepancy β_o .
- * The waves cut the x -axis rapidly when $\alpha = \nu = 1$ and $\beta_o = 0.0$ than when $0 < \alpha < 1$, $0 < \nu < 2$, and $\beta_o = 0.075$.
- * In Figs. 1–7 exhibit the space variation of conductive temperature, thermodynamic temperature, stress, displacement and strain fields at different values of α and ν for two cases of the non-dimensional temperature discrepancy β_o where the case of $\beta_o = 0.0$ indicates the old

Table 1 Values of the constants.

$k = 386 \text{ N/Ks}$, $\alpha_T = 1.78 \times 10^{-5} \text{ K}^{-1}$, $C_E = 383.1 \text{ m}^2/\text{K}$,
$\eta_o = 8886.73 \text{ s/m}^2$,
$\mu = 3.86 \times 10^{10} \text{ N/m}^2$, $\lambda = 7.76 \times 10^{10} \text{ N/m}^2$, $\rho = 8954 \text{ kg/m}^3$,
$c_o = 4158 \text{ m/s}$, $T_o = 293 \text{ K}$, $\varepsilon = 0.0168$, $\tau_o = 0.02 \text{ s}$.

situation (one temperature) and the case of $\beta_o = 0.075$ (Puri and Jordan, 2006), indicates the two-temperature theory.

- * The graphs in Figs. 1–4 represent the variations of conductive temperature ϕ and thermodynamic temperature Θ against the fractional orders α ($0 < \alpha < 1$) and ν ($0 < \nu < 2$). We notice that the particles transport the heat to the other particles easily and this makes the decreasing rate of the temperature greater than the other one (Povstenko, 2005, Sherief et al., 2010 and Youssef, 2010).
- * In Figs. 5–7, we notice that the magnetic field acts to decrease the magnitude of the stress, the displacement and the strain components. This is mainly due to the fact that the magnetic field corresponds to term signifying a positive force that tends to accelerate the charge carriers.
- * Fig. 8 represents the graph of the stress distribution σ against distance x for three models. It is observed that the stress in the three models has a singularity at $x = 0.3$. In the new model for the wide range of α ($0 < \alpha < 1$) and ν ($0 < \nu < 2$), the decreasing rate of the stress is greater than the other one (Youssef, 2010 and Sherief et al., 2010).

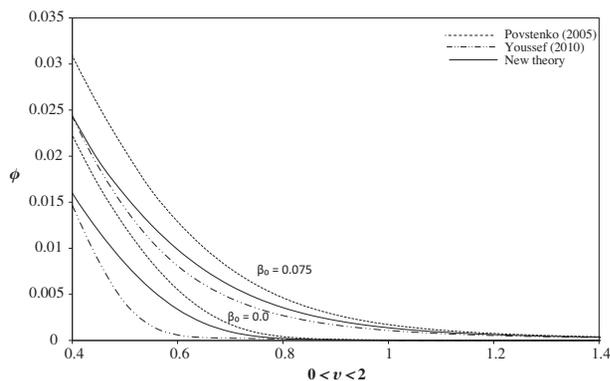


Fig. 1 Dependence of conductive temperature on the integral fractional order ν for different theories.

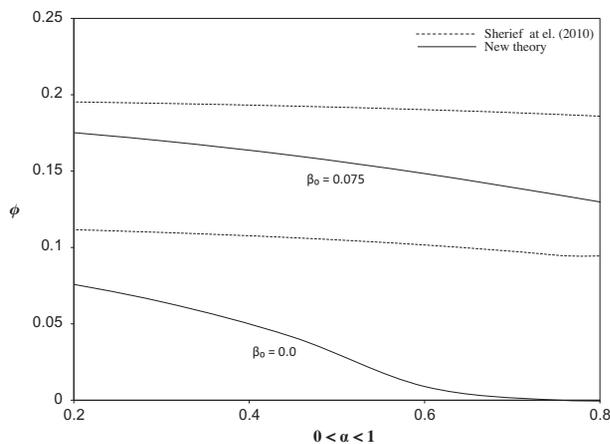


Fig. 2 The conductive temperature distribution against the derivative fractional order α for different theories.

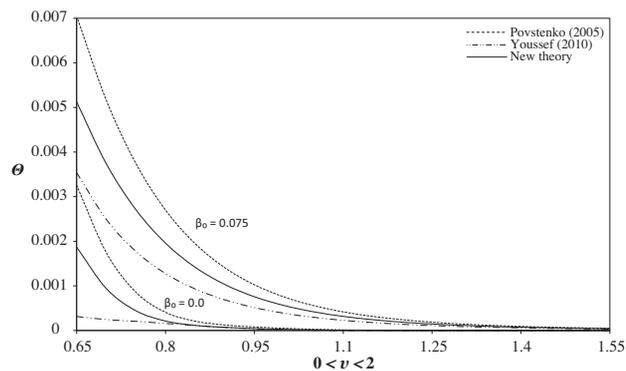


Fig. 3 Dependence of thermodynamic temperature on the integral fractional order ν for different theories.

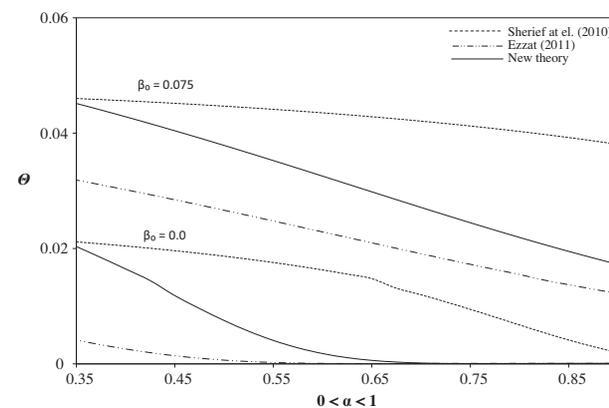


Fig. 4 Dependence of thermodynamic temperature on the derivative fractional order α for different theories.

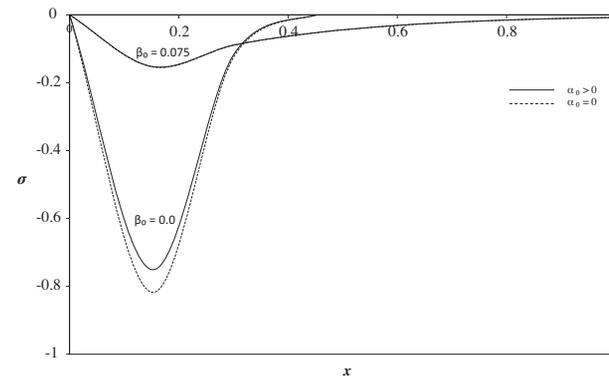


Fig. 5 Dependence of stress distribution on Alfvén velocity and temperature discrepancy.

- * Figs. 9 and 10 represent the distributions of displacement and strain versus the space variable x for three different models (Youssef, 2010, Sherief et al., 2010 and the new model),

which have the same behavior as the thermodynamic temperature distributions except the wide range of x .

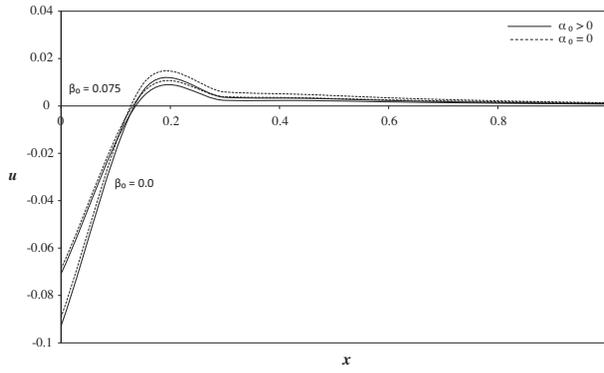


Fig. 6 Dependence of displacement distribution on Alfvén velocity and temperature discrepancy.

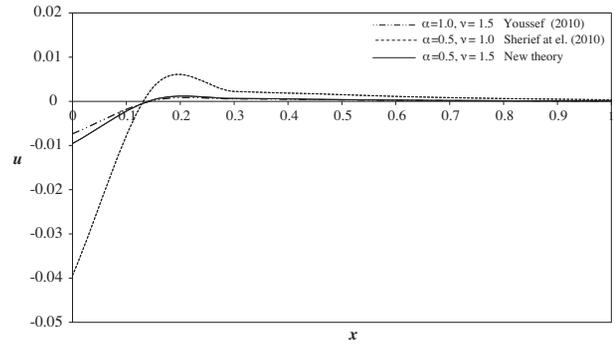


Fig. 9 Dependence of displacement distribution on the derivative and integral fractional orders α and ν for different theories.

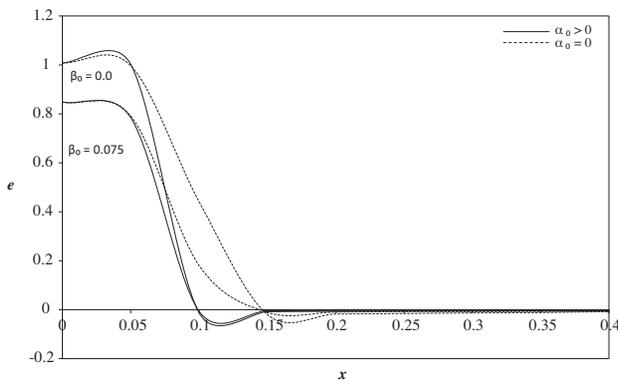


Fig. 7 Dependence of strain distribution on Alfvén velocity and temperature discrepancy.

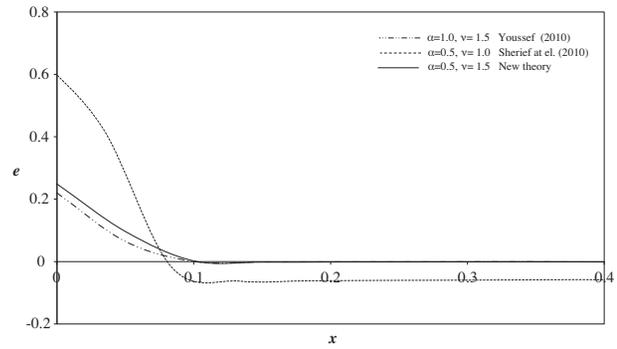


Fig. 10 Dependence of strain distribution on the derivative and integral fractional orders α and ν for different theories.

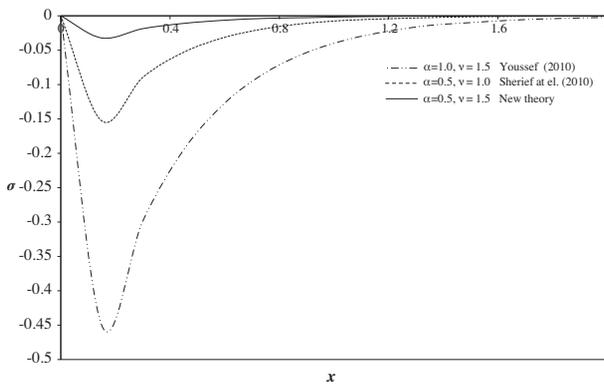


Fig. 8 Dependence of stress distribution on the derivative and integral fractional orders α and ν for different theories.

7. Concluding remarks

* The important phenomenon observed in this problem where the medium is of infinite extent is that the solution of any of the considered function for the new theory vanishes identically outside a bounded region of space. This demonstrates clearly the difference between the coupled and the generalized theories of thermoelasticity. In the first and older theory, the waves propagate with

infinite speeds, so the value of any of the function is not identically zero (though it may be very small) for any large value of x . In the new theory, the response to the thermal and mechanical effects does not reach infinity instantaneously but remains in a bounded region of space given by $0 < x < x^*(t)$.

- * The main goal of this work is to introduce a new mathematical model for Fourier law of heat conduction with time-fractional orders. This model enables us to improve the efficiency of a thermoelectric material figure-of-merit. For a material to be good thermoelectric cooler it must have a high thermoelectric figure of merit, ZT and is defined as, $ZT = \frac{\sigma_0 S^2}{\kappa} T$ (Ezzat, 2011) and it knows that in order to achieve a high thermoelectric material figure-of-merit; one requires a high electrical conductivity (perfect conducting medium) and a low thermal conductivity (indicator fractional orders α and ν).
- * The result provides a motivation to investigate conducting thermoelectric materials as a new class of applicable thermoelectric solids.
- * In this work, the method of direct integration by means of the matrix exponential, which is a standard approach in modern control theory and developed in detail in many texts (Ezzat, 2008), is introduced in the field of electro-magneto-generalized thermoelasticity with fractional heat transfer when the medium is taken as a perfect conductor and is applied to one-dimensional thermal shock problem. The applicability of this approach to the

equations of the two-temperature generalized theory of magneto-thermoelasticity theory is easier than the classical situation (one thermodynamic temperature).

- * The effect of separating the thermodynamic temperature and the conductive temperature is significant in generalized thermoelasticity. The absolute value of the maximum stress decreases relative to the case when the two temperatures coincide. The curves of the stress and temperature distributions are more uniform and the thermodynamic temperature is smaller in magnitude relative to the one-temperature case.

Notations

a	temperature discrepancy
B_i	components of magnetic field strength
c	the speed of light
c_o	$= \sqrt{\frac{\lambda+2\mu}{\rho}}$, speed of propagation of isothermal elastic waves
C_E	specific heat at constant strain
e	dilatation
e_{ij}	components of strain tensor
E_i	components of electric field vector
$H(\cdot)$	Heaviside unit step function.
H_i	magnetic field intensity
J_i	components electric density vector
k	thermal conductivity
q	heat flux vector.
Q	the intensity of applied heat source per unit volume.
t	time
T	absolute thermodynamic temperature.
T_o	$= \varphi_o$ reference temperature
u_i	components of displacement vector
α, ν	fractional orders
α_o	$= \sqrt{\mu_o H_o^2 / \rho}$, Alfven velocity
α_T	coefficient of linear thermal expansion.
α^*	$= 1 + (\alpha_o/c)^2$
β	$= (\alpha_o/c_o)^2$
β_o	dimensionless temperature discrepancy
β_o^c	the critical value of β_o
γ	$= (3\lambda + 2\mu)\alpha_T$
$\delta(\cdot)$	Dirac delta function
δ_{ij}	Kronecker's delta.
ε_{ij}	components of strain tensor.
ε_o	electric permittivity.
ε	thermal coupling parameter
η_o	$= \frac{\rho C_E}{k}$
Θ	$= T - \varphi_o; \quad \left \frac{\Theta}{\varphi_o} \right \ll 1$
λ, μ	Lame' constants
S	Seebeck coefficient
σ_o	electrical conductivity
μ_o	magnetic permeability
ρ	mass density.
σ_{ij}	components of stress tensor.
τ_o	relaxation time.
ϕ	conductive absolute temperature
φ	$= \phi - \varphi_o; \quad \left \frac{\varphi}{\varphi_o} \right \ll 1$

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