Journal of the Association of Arab Universities for Basic and Applied Sciences (2015) **18**, 94–98

University of Bahrain
Journal of the Association of Arab Universities for Basic and Applied Sciences
www.elsevier.com/locate/jaaubas
www.sciencedirect.com

تحويلات ويدر "Widder" التكامليه و تعميماتها في فضاءات بوهيميه"Boehmians"

S. K. Q. Al-Omari

Department of Applied Sciences, Faculty of Engineering Technology, Al-Balqa' Applied University, Amman 11134, Jordan

الملخص:

في هذا البحث، تمت دراسة تحويلات ويدر التكامليه في فضاءات بوهيميه .وقد تم القيام ببناء فضائين مختلفين. كما وجد بعدها بان تحويلات ويدر التكامليه تشكل اقترانات خطيه، متصله بالنسبه لمفهومي الاتصال المعروفين في تلك الفضاءات و متوافقه مع تحويلات ويدر التكامليه الاعتياديه . كما تم التأسيس لبعض الخصائص الاخرى.

Journal of the Association of Arab Universities for Basic and Applied Sciences (2015) **18**, 94–98

University of Bahrain
Journal of the Association of Arab Universities for Basic and Applied Sciences
www.elsevier.com/locate/jaaubas
www.sciencedirect.com

ORIGINAL ARTICLE

On a Widder potential transform and its extension to a space of locally integrable Boehmians

S.K.Q. Al-Omari

Department of Applied Sciences, Faculty of Engineering Technology, Al-Balqa' Applied University, Amman 11134, Jordan

Received 10 January 2014; revised 14 March 2014; accepted 31 March 2014
Available online 24 April 2014

KEYWORDS
Widder potential transform;
Function space;
Generalized function;
Boehmian

Abstract In this paper we investigate a Widder potential transform on certain spaces of Boehmians. We construct two spaces of Boehmians. One space of Boehmians is obtained by a well-known Mellin-type convolution product. The second space is obtained by another mapping acting with the first convolution. The extended Widder potential transform is therefore a mapping, that is, well-defined, linear, continuous, with respect to δ and Δ convergence, and consistent with the classical transform. Certain theorem is also established.

1. Introduction

Let Ω be an open set in $\mathbb{R}^n$ and $f:\Omega\to\mathbb{C}$ be a Lebesgue measurable function on Ω. We denote by $\boldsymbol{l}^{\rho}_{loc}(\Omega)$, $1\leqslant\rho\leqslant\infty$, the complete metrizable space of all f such that for a given $\rho, 1\leqslant\rho\leqslant\infty$, we have

$$\int_{\Bbbk}|f|^{\rho}dx \tag{1}$$

which is finite for all compact subsets $\Bbbk$ of Ω.

For $\rho=1$, it is easy to see that $\boldsymbol{l}^1(\Omega)\subset\boldsymbol{l}^1_{loc}(\Omega)$ and $\boldsymbol{l}^{\rho}_{loc}\subset\boldsymbol{l}^1_{loc}$, where $\boldsymbol{l}^1(\Omega)$ is the set of globally integrable functions, $1<\rho\leqslant\infty$.

It may also be noted that every continuous function is a locally integrable function and, for all $f,g\in\boldsymbol{l}^{\rho}_{loc}$, $1<\rho\leqslant\infty$, $f+g$ and αf are also in $\boldsymbol{l}^{\rho}_{loc}$, where $\alpha\in\mathbb{C}$, $\mathbb{C}$ being the field of complex numbers.

Denote by $(0,\infty)$ the set of positive real numbers. The Widder potential transform was presented by Widder (1966, 1971), by the integral equation

$$(\mathcal{P}f)(y)=\int_0^{\infty}\frac{x}{x^2+y^2}f(x)dx \tag{2}$$

as a transform related to the Poison integral of a harmonic function in a half plane. The Parseval–Goldstein type formula of the Widder potential transform was given by Srivastava and Singh (1985), as follows

$$\int_0^{\infty}x(\mathcal{P}f)(x)g(x)dx=\int_0^{\infty}xf(x)(\mathcal{P}g)(x)dx.$$

The transform under consideration and its Parseval–Goldstein type theorem involving the classical Laplace and Fourier sine are established by Srivastava and Yürekli (1991). More about the Widder potential and Laplace-type transforms and the Parseval–Goldstein type theorem reader can see Yürekli and Sadek (1991) and Dernek et al. (2011).

E-mail address: s.k.q.alomari@fet.edu.jo
Peer review under responsibility of University of Bahrain.

1815-3852 © 2014 Production and hosting by Elsevier B.V. on behalf of University of Bahrain.
http://dx.doi.org/10.1016/j.jaubas.2014.03.004

2. General Boehmians

The construction of Boehmians consists of the following elements:

(i) A set $\boldsymbol{A}$;
(ii) A commutative semigroup $(\boldsymbol{B}, *)$;
(iii) An operation $\odot : \boldsymbol{A} \times \boldsymbol{B} \to \boldsymbol{A}$ such that for each $x \in \boldsymbol{A}$ and $\upsilon_1, \upsilon_2, \in \boldsymbol{B}$,

$$x \odot (\upsilon_1 * \upsilon_2) = (x \odot \upsilon_1) \odot \upsilon_2;$$

(iv) A set $\Delta \subset \boldsymbol{B}^{\mathbb{N}}$ satisfying:
(a) If $x, y \in \boldsymbol{A}, (\upsilon_n) \in \Delta, x \odot \upsilon_n = y \odot \upsilon_n$ for all n, then $x = y$;
(b) If $(\upsilon_n), (\sigma_n) \in \Delta$, then $(\upsilon_n * \sigma_n) \in \Delta.\Delta$ is the set of all delta sequences.

Consider

$$\mathcal{A} = \{(x_n, \upsilon_n) : x_n \in \mathbf{A}, (\upsilon_n) \in \Delta, x_n \odot \upsilon_m = x_m \odot \upsilon_n, \forall m, n \in \mathbb{N}\}.$$

If $(x_n, \upsilon_n), (y_n, \sigma_n) \in \mathcal{A}, x_n \odot \sigma_m = y_m \odot \upsilon_n, \forall m, n \in \mathbb{N}$, then we say $(x_n, \upsilon_n) \sim (y_n, \sigma_n)$. The relation $\sim$ is an equivalence relation in $\mathcal{A}$. The space of equivalence classes in $\mathcal{A}$ is denoted by $\boldsymbol{\kappa}(\boldsymbol{A}, (\boldsymbol{B}, *), \odot, \Delta)$. Elements of $\boldsymbol{\kappa}(\boldsymbol{A}, (\boldsymbol{B}, *), \odot, \Delta)$ are called Boehmians.

Between $\boldsymbol{A}$ and $\boldsymbol{\kappa}(\boldsymbol{A}, (\boldsymbol{B}, *), \odot, \Delta)$ there is a canonical embedding expressed as

$$x \to \frac{x \odot s_n}{s_n} \quad \text{as } n \to \infty.$$

The operation $\odot$ can be extended to $\boldsymbol{\kappa}(\boldsymbol{A}, (\boldsymbol{B}, *), \odot, \Delta) \times \boldsymbol{A}$ by

$$\frac{x_n}{\upsilon_n} \odot t = \frac{x_n \odot t}{\upsilon_n}.$$

In $\boldsymbol{\kappa}(\boldsymbol{A}, (\boldsymbol{B}, *), \odot, \Delta)$, two types of convergence are:

1. A sequence $(h_n) \in \boldsymbol{\kappa}(\boldsymbol{A}, (\boldsymbol{B}, *), \odot, \Delta)$ is said to be δ convergent to $h \in \boldsymbol{\kappa}(\boldsymbol{A}, (\boldsymbol{B}, *), \odot, \Delta)$, denoted by $h_n \xrightarrow{\delta} h$ as $n \to \infty$, if there exists a delta sequence (υ_n) such that $(h_n \odot \upsilon_n)$, $(h \odot \upsilon_n) \in \boldsymbol{A}, \forall k, n \in \mathbb{N}$, and $(h_n \odot \upsilon_k) \to (h \odot \upsilon_k)$ as $n \to \infty$, in $\boldsymbol{A}$, for every $k \in \mathbb{N}$.
2. A sequence $(h_n) \in \boldsymbol{\kappa}(\boldsymbol{A}, (\boldsymbol{B}, *), \odot, \Delta)$ is said to be Δ convergent to $h \in \boldsymbol{\kappa}(\boldsymbol{A}, (\boldsymbol{B}, *), \odot, \Delta)$, denoted by $h_n \xrightarrow{\Delta} h$ as $n \to \infty$, if there exists a $(\upsilon_n) \in \Delta$ such that $(h_n - h) \odot \upsilon_n \in \boldsymbol{A}$, $\forall n \in \mathbb{N}$, and $(h_n - h) \odot \upsilon_n \to 0$ as $n \to \infty$ in $\boldsymbol{A}$.

The following theorem is equivalent to the statement of δ convergence:

Remark 1. $h_n \xrightarrow{\delta} h (n \to \infty) \in \boldsymbol{\kappa}(\boldsymbol{A}, (\boldsymbol{B}, *), \odot, \Delta)$ if and only if there is $f_{n,k}, f_k \in \boldsymbol{A}$ and $\upsilon_k \in \Delta$ such that $h_n = \left[\frac{f_{n,k}}{\upsilon_k}\right], h = \left[\frac{f_k}{\upsilon_k}\right]$ and for each $k \in \mathbb{N}, f_{n,k} \to f_k$ as $n \to \infty$ in $\boldsymbol{A}$.

For more details we refer to Al-Omari and Kilicman (2012a,b, 2013), Al-Omari (2013a,b,c), Beardsley and Mikusinski (2013), Bhuvaneswari and Karunakaran (2010), Boehme (1973), Ganesan (2010), Karunakaran and Ganesan (2009), Karunakaran and Angeline Chella (2011), Loonker and Banerji (2010), Loonker et al. (2010), Mikusinski (1987, 1983, 1995), Nemzer (2010, 2007), Roopkumar (2009), Srivastava and Singh (1985) and Roopkumar (2009), and many others.

3. Constructed spaces of Boehmians

In this section we construct the spaces $\boldsymbol{\delta}(\boldsymbol{l}^{\rho}_{loc}, (\boldsymbol{\kappa}, \bullet), \bullet, \Delta)$ and $\boldsymbol{\delta}(\boldsymbol{l}^{\rho}_{loc}, (\boldsymbol{\kappa}, \bullet), *, \Delta)$ of Boehmians.

Following theorem is straightforward.

Theorem 1. *Let $f \in \boldsymbol{l}^{\rho}_{loc}, 1 \leqslant \rho < \infty$; then we have $\mathcal{P}f \in \boldsymbol{l}^{\rho}_{loc}$.*

In (3) and (4), two operations are needful for our construction.

The Mellin-type convolution between two functions f and φ is defined by Zemanian (1987) [25],

$$(f \bullet \varphi)(x) = \int_0^{\infty} f(xt^{-1}) t^{-1} \varphi(t) dt. \tag{3}$$

More properties that $\bullet$ enjoys can be found in the above citation.

On the other hand, denote by $*$ the product given by

$$(f * \varphi)(y) = \int_0^{\infty} f(yt^{-1}) t^{-1} \varphi(t) dt. \tag{4}$$

Following is a theorem which is essential in the sense of our results.

Theorem 2. *Let $f \in \boldsymbol{l}^{\rho}_{loc}, 1 \leqslant \rho < \infty, \varphi \in \boldsymbol{\kappa}(0, \infty)$; then we have*

$$\mathcal{P}(f \bullet \varphi)(y) = (\mathcal{P}f * \varphi)(y).$$

Proof. Let $f \in \boldsymbol{l}^{\rho}_{loc}$ and $\varphi \in \boldsymbol{\kappa}(0, \infty), 1 \leqslant \rho < \infty$, be given then, using (3) we get

$$\begin{aligned} \mathcal{P}(f \bullet \varphi)(y) &= \int_0^{\infty} \frac{x}{x^2 + y^2} (f \bullet \varphi)(x) dx \\ &= \int_0^{\infty} \frac{x}{x^2 + y^2} \left(\int_0^{\infty} f(xt^{-1}) t^{-1} \varphi(t) dt \right) dx. \end{aligned} \tag{5}$$

The change of variables $x = tz$ transforms (5) into

$$\begin{aligned} \mathcal{P}(f \bullet \varphi)(y) &= \int_0^{\infty} \int_0^{\infty} \frac{z f(z)}{z^2 + \left(\frac{y}{t}\right)^2} dz t^{-1} g(t) dt \\ &= \int_0^{\infty} (\mathcal{P}f)(yt^{-1}) t^{-1} g(t) dt. \end{aligned}$$

Hence, by (4), we get

$$\mathcal{P}(f \bullet \varphi)(y) = (\mathcal{P}f * \varphi)(y)$$

This completes the proof of the theorem. □

Proof of the following two theorems is straightforward.

Theorem 3. *Let $f \in \boldsymbol{l}^{\rho}_{loc}, \varphi, \psi \in \boldsymbol{\kappa}(0, \infty), 1 \leqslant \rho < \infty$; then we have*

(i) $f \bullet (\varphi + \psi) = f \bullet \varphi + f \bullet \psi$.
(ii) $(\alpha f) \bullet \varphi = \alpha (f \bullet \varphi), \alpha \in \mathbb{C}$.

Theorem 4. *Let $f_n \to f$ in $\boldsymbol{l}^{\rho}_{loc}, 1 \leqslant \rho < \infty$, as $n \to \infty$, and $\varphi \in \boldsymbol{\kappa}(0, \infty)$; then $f_n \bullet \varphi \to f \bullet \varphi$ as $n \to \infty$.*

This theorem follows from the properties of integration.

Theorem 5. *Let $f \in \boldsymbol{l}^{\rho}_{loc}, 1 \leqslant \rho < \infty, \varphi, \psi \in \boldsymbol{\kappa}(0, \infty)$; then $f \bullet (\varphi \bullet \psi) = (f \bullet \varphi) \bullet \psi$.*

Similar proof to this theorem is given by Zemanian (1987). Hence, we prefer omiting the details.

Denote by Δ the set of delta sequences $(\delta_n) \in \boldsymbol{\kappa}(0, \infty)$ satisfying

$$\int_0^\infty \delta_n(x)dx = 1. \tag{6}$$

$$\int_0^\infty \mid \delta_n(x) \mid dx < M, \quad 0 < M \in (0, \infty). \tag{7}$$

$$\text{supp}\delta_n(x) \subset (0, \varepsilon_n), \varepsilon_n \to 0 \quad \text{as } n \to \infty. \tag{8}$$

Theorem 6. *Let $(\delta_n) \in \Delta$ and $f \in \boldsymbol{l}^\rho_{loc}, 1 \leqslant \rho < \infty$; then $f \bullet \delta_n \to f$ as $n \to \infty$.*

Proof. Let $(\delta_n) \in \Delta, f \in \boldsymbol{l}^\rho_{loc}, 1 \leqslant \rho < \infty$ and $\Bbbk$ be a compact subset of $(0, \infty)$ containing suppδ_n for all n; then using (6) we get

$$\int_\Bbbk \mid (f \bullet \delta_n - f)(x) \mid dx = \int_\Bbbk \left| \int_0^\infty (f \bullet \delta_n)(x) - f(x) \int_0^\infty \delta_n(y)dy \right| dx$$
$$\leqslant \int_\Bbbk \int_0^\infty \left| f\left(\frac{x}{y}\right) y^{-1} - f(x) \right| \mid \delta_n(y) \mid dydx.$$

By Fubinitz theorem we get

$$\int_\Bbbk \mid (f \bullet \delta_n - f)(x) \mid dx \leqslant \int_0^\infty \int_\Bbbk \left| f\left(\frac{x}{y}\right) y^{-1} - f(x) \right| dx$$
$$\mid \delta_n(y) \mid dy. \tag{9}$$

Since $f\left(\frac{x}{y}\right) y^{-1} \in \boldsymbol{l}^\rho_{loc}$ it follows that $f\left(\frac{x}{y}\right) y^{-1} - f(x) \in \boldsymbol{l}^\rho_{loc}$.

Hence, (9) implies

$$\int_\Bbbk \mid (f \bullet \delta_n - f)(x) \mid dx \leqslant M_1 \int_\Bbbk \mid \delta_n(y) \mid dy \tag{10}$$

where $M_1 \in (0, \infty)$.

Therefore, **if** $\Bbbk = [a, b], a, b > 0$; then it follows from (8) that

$$\int_\Bbbk \mid (f \bullet \delta_n - f)(x) \mid dx \leqslant M_1 \varepsilon_n (b - a) \to 0 \quad \text{as } n \to \infty. \tag{11}$$

Hence

$$\int_\Bbbk \mid (f \bullet \delta_n - f)(x) \mid = \int_\Bbbk \mid (f \bullet \delta_n)(x) \mid dx$$
$$- \int_\Bbbk \mid f(x) \mid dx \to 0 \quad \text{as } n \to \infty.$$

Therefore $f \bullet \delta_n \to f$ as $n \to \infty$ on compact subsets of $(0, \infty)$.

This completes the proof of the theorem. □

The Boehmian space $\boldsymbol{\delta}(\boldsymbol{l}^\rho_{loc}, (\boldsymbol{\kappa}, \bullet), \bullet, \Delta)$ is therefore described.

We next establish the space $\boldsymbol{\delta}(\boldsymbol{l}^\rho_{loc}, (\boldsymbol{\kappa}, \bullet), *, \Delta)$.

Theorem 7. *Let $f \in \boldsymbol{l}^\rho_{loc}$ and $\varphi \in \boldsymbol{\kappa}(0, \infty)$; then we have $f * \varphi \in \boldsymbol{\kappa}(0, \infty)$.*

Theorem 8. *Let $f \in \boldsymbol{l}^\rho_{loc}, \varphi, \psi \in \boldsymbol{\kappa}(0, \infty), 1 \leqslant \rho < \infty$; then we have*

(i) $f * (\varphi + \psi) = f * \varphi + f * \psi$
(ii) $(\alpha f) * \varphi = \alpha(f * \varphi), \alpha \in \mathbb{C}$.

Theorem 9.

(i) *Let $f_n \to f$ in $\boldsymbol{l}^\rho_{loc}, 1 \leqslant \rho < \infty$, as $n \to \infty$, and $\varphi \in \boldsymbol{\kappa}(0, \infty)$; then $f_n * \varphi \to f * \varphi$ as $n \to \infty$.*
(ii) *Let $(\delta_n) \in \Delta, f \in \boldsymbol{l}^\rho_{loc}, 1 \leqslant \rho < \infty$; then $f * \delta_n \to f$ as $n \to \infty$.*

For similar proofs to Theorems 7–9, see Theorems 3, 4 and 6.

As a next step, we merely need to establish the following theorem.

Theorem 10. *Let $f \in \boldsymbol{l}^\rho_{loc}, 1 \leqslant \rho < \infty, \varphi, \psi \in \boldsymbol{\kappa}(0, \infty)$; then $f * (f \bullet \psi) = (f * \varphi) * \psi$.*

Proof. Let $f \in \boldsymbol{l}^\rho_{loc}, \varphi, \psi \in \boldsymbol{\kappa}(0, \infty)$; then, using (3) and (4) we write

$$(f * (f \bullet \psi))(y) = \int_0^\infty f(yt^{-1})t^{-1}(\varphi \bullet \psi)(t)dt$$
$$= \int_0^\infty f(yt^{-1})t^{-1}\left(\int_0^\infty \varphi(tx^{-1})x^{-1}\psi(x)dx\right)dt$$
$$= \int_0^\infty \left(\int_0^\infty f(yt^{-1})\varphi(tx^{-1})t^{-1}dt\right)x^{-1}\psi(x)dx. \tag{12}$$

The substitution $tx^{-1} = z$ implies $dt = xdz$ and hence, from (12), we have

$$(f * (f \bullet \psi))(y) = \int_0^\infty \left(\int_0^\infty f(yx^{-1}z^{-1})z^{-1}\varphi(z)dz\right)x^{-1}\psi(x)dx$$
$$= \int_0^\infty (f * \varphi)(yx^{-1})x^{-1}\psi(x)dx.$$

Therefore

$$(f * (\varphi \bullet \psi))(y) = ((f * \varphi) * \psi)(y).$$

This completes the proof of the theorem. □

Thus our Boehmian space $\boldsymbol{\delta}(\boldsymbol{l}^\rho_{loc}, (\boldsymbol{\kappa}, \bullet), *, \Delta)$ is recognized.

4. The generalized widder potential transform

Let $\left[\frac{(f_n)}{(\delta_n)}\right] \in \boldsymbol{\delta}(\boldsymbol{l}^\rho_{loc}, (\boldsymbol{\kappa}, \bullet), \bullet, \Delta)$ be given then we define its extended Widder transform as follows

$$\boldsymbol{P}_e\left(\left[\frac{(f_n)}{(\delta_n)}\right]\right) = \left[\frac{(\mathcal{P}f_n)}{(\delta_n)}\right] \tag{13}$$

in $\boldsymbol{\delta}(\boldsymbol{l}^\rho_{loc}, (\boldsymbol{\kappa}, \bullet), *, \Delta)$.

Theorem 11. *$\boldsymbol{P}_e : \boldsymbol{\delta}(\boldsymbol{l}^\rho_{loc}, (\boldsymbol{\kappa}, \bullet), \bullet, \Delta) \to \boldsymbol{\delta}(\boldsymbol{l}^\rho_{loc}, (\boldsymbol{\kappa}, \bullet), *, \Delta)$ is a well-defined and linear extension of $\mathcal{P}$.*

Proof. Let $\left[\frac{(\alpha_n)}{(r_n)}\right] = \left[\frac{(v_n)}{(\psi_n)}\right] \in \boldsymbol{\delta}(\boldsymbol{l}^\rho_{loc}, (\boldsymbol{\kappa}, \bullet), \bullet, \Delta)$; then $\alpha_n \bullet \psi_m = v_m \bullet r_n = v_n \bullet r_m$. Employing the potential transform $\mathcal{P}$ on both sides implies $\mathcal{P}\alpha_n * \psi_m = \mathcal{P}v_n * r_m, \forall n, m$. Thus, $\frac{\mathcal{P}\alpha_n}{r_n}$ are $\frac{\mathcal{P}v_n}{\psi_n}$ equivalent. Therefore, $\left[\frac{(\mathcal{P}\alpha_n)}{(r_n)}\right] = \left[\frac{(\mathcal{P}v_n)}{(\psi_n)}\right]$.

To proof the second part of the theorem, let $\left[\frac{(\alpha_n)}{(r_n)}\right], \left[\frac{(v_n)}{(\psi_n)}\right] \in \boldsymbol{\delta}(\boldsymbol{l}^\rho_{loc}, (\boldsymbol{\kappa}, \bullet), \bullet, \Delta)$ then

$$\boldsymbol{P}_e\left(\left[\frac{(\alpha_n)}{(r_n)}\right]+\left[\frac{(v_n)}{(\psi_n)}\right]\right)=\boldsymbol{P}_e\left(\left[\frac{(\alpha_n)\bullet(\psi_n)+(v_n)\bullet(r_n)}{(r_n)\bullet(\psi_n)}\right]\right)$$

$$\text{i.e.}\quad =\left[\frac{\mathcal{P}((\alpha_n)\bullet(\psi_n)+(v_n)\bullet(r_n))}{(r_n)*(\psi_n)}\right]$$

$$\text{i.e.}\quad =\left[\frac{(\mathcal{P}\alpha_n)*(\psi_n)+(\mathcal{P}v_n)*(r_n)}{(r_n)*(\psi_n)}\right]$$

Hence, we get

$$\boldsymbol{P}_e(\left[\frac{(\alpha_n)}{(r_n)}\right]+\left[\frac{(v_n)}{(\psi_n)}\right])=\boldsymbol{P}_e\left[\frac{(\alpha_n)}{(r_n)}\right]+\boldsymbol{P}_e\left[\frac{(v_n)}{(\psi_n)}\right].$$

Let $\lambda\in\mathbb{C}$ then $\lambda\boldsymbol{P}_e\left[\frac{(\alpha_n)}{(r_n)}\right]=\lambda\left[\frac{(\mathcal{P}\alpha_n)}{(r_n)}\right]=\left[\frac{\mathcal{P}(\lambda\alpha_n)}{(r_n)}\right]$. Hence

$$\lambda\boldsymbol{P}_e\left[\frac{(\alpha_n)}{(r_n)}\right]=\boldsymbol{P}_e\left(\lambda\left[\frac{(\alpha_n)}{(r_n)}\right]\right).$$

This completes the proof. □

Definition 12. Let $\left[\frac{(\mathcal{P}f_n)}{(\phi_n)}\right]\in\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),*,\Delta)$; then we define the inverse $\boldsymbol{P}_e$ transform of $\left[\frac{(\mathcal{P}f_n)}{(\phi_n)}\right]$ as follows

$$\boldsymbol{P}_e^{-1}\left[\frac{(\mathcal{P}f_n)}{(\phi_n)}\right]=\left[\frac{(\mathcal{P})^{-1}(\mathcal{P}f_n)}{(\phi_n)}\right]=\left[\frac{(f_n)}{(\phi_n)}\right]\tag{14}$$

for each $(\phi_n)\in\Delta$.

Theorem 13. *$\boldsymbol{P}_e^{-1}$ is well-defined and linear.*

Proof of this theorem is analogous to that of above theorem.

Theorem 14. *$\boldsymbol{P}_e:\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),\bullet,\Delta)\to\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),*,\Delta)$ is a one-one and onto extension of $\mathcal{P}$.*

Proof. Let $\boldsymbol{P}_e\left[\frac{(f_n)}{(\phi_n)}\right]=\boldsymbol{P}_e\left[\frac{(g_n)}{(\psi_n)}\right]$. Upon using (13) and the concept of quotients in $\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),*,\Delta)$ implies $\mathcal{P}f_n*\psi_m=\mathcal{P}g_m*\phi_n$, $\forall m,n\in\mathbb{N}$. Theorem 2 then implies $\mathcal{P}(f_n\bullet\psi_m)=\mathcal{P}(g_m\bullet\phi_n)$, $\forall m,n\in\mathbb{N}$. Hence $f_n\bullet\psi_m=g_m\bullet\phi_n$ and therefore

$$\left[\frac{(f_n)}{(\phi_n)}\right]=\left[\frac{(g_n)}{(\psi_n)}\right].$$

To establish that $\mathcal{P}$ is surjective, let $\left[\frac{(\mathcal{P}f_n)}{(\phi_n)}\right]\in\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),*,\Delta)$. Then we get $\mathcal{P}f_n*\phi_m=\mathcal{P}f_m*\phi_n,\forall m,n\in\mathbb{N}$. Once again, Theorem 2 implies $\mathcal{P}(f_n\bullet\phi_m)=\mathcal{P}(f_m\bullet\phi_n)$. Therefore, $\left[\frac{(f_n)}{(\phi_n)}\right]\in\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),\bullet,\Delta)$ is such that

$$\boldsymbol{P}_e\left[\frac{(f_n)}{(\phi_n)}\right]=\left[\frac{(\mathcal{P}f_n)}{(\phi_n)}\right].$$

This completes the proof of the theorem. □

Theorem 15. *Let $\left[\frac{(\mathcal{P}f_n)}{(\phi_n)}\right]\in\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),*,\Delta)$ and $\phi\in\boldsymbol{\kappa}(0,\infty)$; then*

$$\boldsymbol{P}_e^{-1}\left(\left[\frac{(\mathcal{P}f_n)}{(\phi_n)}\right]*\phi\right)=\left[\frac{(f_n)}{(\phi_n)}\right]\bullet\phi$$

and

$$\boldsymbol{P}_e\left(\left[\frac{(f_n)}{(\phi_n)}\right]\bullet\phi\right)=\left[\frac{(\mathcal{P}f_n)}{(\phi_n)}\right]*\phi.$$

Detailed proof of the first part is as follows:

Applying (14) yields

$$\boldsymbol{P}_e^{-1}\left(\left[\frac{(\mathcal{P}f_n)}{(\phi_n)}\right]*\phi\right)=\boldsymbol{P}_e^{-1}\left(\left[\frac{(\mathcal{P}f_n)*\phi}{(\phi_n)}\right]\right)=\left[\frac{(\mathcal{P})^{-1}((\mathcal{P}f_n)*\phi)}{(\phi_n)}\right].$$

Using Theorem 2 we obtain

$$\boldsymbol{P}_e^{-1}\left(\left[\frac{(\mathcal{P}f_n)}{(\phi_n)}\right]*\phi\right)=\left[\frac{(f_n)\bullet\phi}{(\phi_n)}\right]=\left[\frac{(f_n)}{(\phi_n)}\right]\bullet\phi.$$

The proof of the part that $\boldsymbol{P}_e\left(\left[\frac{(f_n)}{(\phi_n)}\right]\bullet\phi\right)=\left[\frac{(\mathcal{P}f_n)}{(\phi_n)}\right]*\phi$ is similar.

This completes the proof of the theorem.

Theorem 16. *$\boldsymbol{P}_e:\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),\bullet,\Delta)\to\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),*,\Delta)$ and $\boldsymbol{P}_e^{-1}:\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),*,\Delta)\to\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),\bullet,\Delta)$ are continuous with respect to δ and Δ convergence.*

Proof. First of all, we show that $\boldsymbol{P}_e:\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),\bullet,\Delta)\to\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),*,\Delta)$ and $\boldsymbol{P}_e^{-1}:\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),*,\Delta)\to\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),\bullet,\Delta)$ are continuous with respect to δ convergence.

Let $\beta_n\xrightarrow{\delta}\beta$ in $\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),\bullet,\Delta)$ as $n\to\infty$ then we show that $\boldsymbol{P}_e\beta_n\to\boldsymbol{P}_e\beta$ as $n\to\infty$. By virtue of Remark 1, we can find $f_{n,k}$ and f_k in $\boldsymbol{l}^p_{loc}$ such that $\beta_n=\left[\frac{f_{n,k}}{\phi_k}\right]$ and $\beta=\left[\frac{f_k}{\phi_k}\right]$ and $f_{n,k}\to f_k$ as $n\to\infty$ for every $k\in\mathbb{N}$. Employing the continuity condition of $\mathcal{P}$ transform implies $\mathcal{P}f_{n,k}\to\mathcal{P}f_k$ as $n\to\infty$ in the space $\boldsymbol{l}^p_{loc}$. Thus, $\left[\frac{\mathcal{P}f_{n,k}}{\phi_k}\right]\to\left[\frac{\mathcal{P}f_k}{\phi_k}\right]$ as $n\to\infty$ in $\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),*,\Delta)$.

To prove the second part, let $g_n\xrightarrow{\delta}g$ in $\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),*,\Delta)$ as $n\to\infty$. Once again, by Remark 1, there are $g_n=\left[\frac{\mathcal{P}f_{n,k}}{\phi_k}\right]$ and $g=\left[\frac{\mathcal{P}f_k}{\phi_k}\right]$ and $\mathcal{P}f_{n,k}\to\mathcal{P}f_k$ as $n\to\infty$. Hence $f_{n,k}\to f_k$ in $\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),\bullet,\Delta)$ as $n\to\infty$. That is, $\left[\frac{f_{n,k}}{\phi_k}\right]\to\left[\frac{f_k}{\phi_k}\right]$ as $n\to\infty$.

Using (14) we get $\boldsymbol{P}_e^{-1}\left[\frac{\mathcal{P}f_{n,k}}{\phi_k}\right]\to\boldsymbol{P}_e^{-1}\left[\frac{\mathcal{P}f_k}{\phi_k}\right]$ as $n\to\infty$.

Now, we establish the continuity of $\boldsymbol{P}_e$ and $\boldsymbol{P}_e^{-1}$ with respect to Δ convergence:

Let $\beta_n\xrightarrow{\Delta}\beta$ in $\boldsymbol{\delta}(\boldsymbol{l}^p_{loc},(\boldsymbol{\kappa},\bullet),\bullet,\Delta)$ as $n\to\infty$. Then, there can be found $(f_n)\in\boldsymbol{l}^p_{loc}$ and $(\phi_n)\in\Delta$ such that $(\beta_n-\beta)\bullet\phi_n=\left[\frac{(f_n)\bullet\phi_k}{\phi_k}\right]$ and $f_n\to0$ as $n\to\infty$.

Employing (13) we get $\boldsymbol{P}_e((\beta_n-\beta)\bullet\phi_n)=\left[\frac{\mathcal{P}((f_n)\bullet\phi_k)}{\phi_k}\right]$. Hence, we have $\boldsymbol{P}_e((\beta_n-\beta)\bullet\phi_n)=\left[\frac{(\mathcal{P}f_n)*\phi_k}{\phi_k}\right]=\mathcal{P}f_n\to0$ as $n\to\infty$ in $\boldsymbol{l}^p_{loc}$.

Therefore $\boldsymbol{P}_e((\beta_n-\beta)\bullet\phi_n)=(\boldsymbol{P}_e\beta_n-\boldsymbol{P}_e\beta)*\phi_n\to0$ as $n\to\infty$. Hence, $\boldsymbol{P}_e\beta_n\xrightarrow{\Delta}\boldsymbol{P}_e\beta$ as $n\to\infty$.

Finally, let $g_n \xrightarrow{\Delta} g \in \boldsymbol{\delta}(\boldsymbol{l}^{\rho}_{loc}, (\boldsymbol{\kappa}, \bullet), *, \Delta)$ as $n \to \infty$ then we find $\mathcal{P}f_k \in \boldsymbol{l}^{\rho}_{loc}$ such that $(g_n - g) * \phi_n = \left[\frac{\mathcal{P}f_k * \phi_k}{\phi_k}\right]$ and $\mathcal{P}f_k \to 0$ as $n \to \infty$ for some $(\phi_n) \in \Delta$. Using (14), we get

$$\boldsymbol{P}_e^{-1}((g_n - g) * \phi_n) = \left[\frac{(\mathcal{P})^{-1}(\mathcal{P}f_k * \phi_k)}{\phi_k}\right].$$

Theorem 2 implies $\boldsymbol{P}_e^{-1}((g_n - g) * \phi_n) = \left[\frac{(f_n) \bullet \phi_k}{\phi_k}\right] = f_n \to 0$ as $n \to \infty$ in $\boldsymbol{l}^{\rho}_{loc}$. Thus

$$\boldsymbol{P}_e^{-1}((g_n - g) * \phi_n) = \left(\boldsymbol{P}_e^{-1} g_n - \boldsymbol{P}_e^{-1} g\right) \bullet \phi_n \to 0 \quad \text{as } n \to \infty.$$

From this we find that $\boldsymbol{P}_e^{-1} g_n \xrightarrow{\Delta} \boldsymbol{P}_e^{-1} g$ as $n \to \infty$ in $\boldsymbol{\delta}(\boldsymbol{l}^{\rho}_{loc}, (\boldsymbol{\kappa}, \bullet), \bullet, \Delta)$.

This completes the proof of the theorem. □

Theorem 17. *The extended* $\boldsymbol{P}_e$ *transform is consistent with* $\mathcal{P}(\mathcal{P} : \boldsymbol{l}^{\rho}_{loc} \to \boldsymbol{l}^{\rho}_{loc})$.

Proof. For every $f \in \boldsymbol{l}^{\rho}_{loc}$, let β be its representative in $\boldsymbol{\delta}(\boldsymbol{l}^{\rho}_{loc}, (\boldsymbol{\kappa}, \bullet), \bullet, \Delta)$; then $\beta = \left[\frac{f \bullet (\varphi_n)}{(\varphi_n)}\right]$, where $(\varphi_n) \in \Delta, \forall n \in \mathbb{N}$. Its clear that (φ_n) is independent from the representative, $\forall n \in \mathbb{N}$. Therefore

$$\boldsymbol{P}_e(\beta) = \boldsymbol{P}_e\left(\left[\frac{f \bullet (\varphi_n)}{(\varphi_n)}\right]\right) = \left[\frac{\mathcal{P}(f \bullet (\varphi_n))}{(\varphi_n)}\right] = \left[\frac{\mathcal{P}f * (\varphi_n)}{(\varphi_n)}\right]$$

which is the representative of $\mathcal{P}f \in \boldsymbol{l}^{\rho}_{loc}$.

Hence the proof. □

Theorem 18. *Let* $\beta = \left[\frac{(f_n)}{(\varphi_n)}\right] \in \boldsymbol{\delta}(\boldsymbol{l}^{\rho}_{loc}, (\boldsymbol{\kappa}, \bullet), \bullet, \Delta)$ *and* $\gamma = \left[\frac{(\kappa_n)}{(\phi_n)}\right] \in \boldsymbol{\delta}(\boldsymbol{l}^{\rho}_{loc}, (\boldsymbol{\kappa}, \bullet), \bullet, \Delta)$*; then*

$$\boldsymbol{P}_e(\beta \bullet \gamma) = \boldsymbol{P}_e \beta * \gamma.$$

Proof. Assume the requirements of the theorem are satisfied for some β and $\gamma \in \boldsymbol{\delta}(\boldsymbol{l}^{\rho}_{loc}, (\boldsymbol{\kappa}, \bullet), \bullet, \Delta)$ then we indeed get

$$\boldsymbol{P}_e(\beta \bullet \gamma) = \boldsymbol{P}_e\left(\left[\frac{(f_n) \bullet (\kappa_n)}{(\varphi_n) \bullet (\phi_n)}\right]\right) = \left[\frac{\mathcal{P}((f_n) \bullet (\kappa_n))}{(\varphi_n) \bullet (\phi_n)}\right].$$

Hence

$$\boldsymbol{P}_e(\beta \bullet \gamma) = \left[\frac{(\mathcal{P}f_n) * (\kappa_n)}{(\varphi_n) * (\phi_n)}\right] = \left[\frac{(\mathcal{P}f_n)}{(\varphi_n)}\right] * \left[\frac{(\kappa_n)}{(\phi_n)}\right].$$

This completes the proof of the theorem. □

References

Al-Omari, S.K.Q., 2013a. Distributional and tempered distributional diffraction Fresnel transforms and their extension to Boehmian spaces. Italian J. Pure Appl. Math. (30), 179–194.

Al-Omari, S.K.Q., 2013b. On the application of Natural transforms. Int. J. Pure Appl. Math. 85 (4), 729–744.

Al-Omari, S.K.Q., 2013c. Hartley transforms on certain space of generalized functions. Georgian Math. J. 20 (3), 415–426.

Al-Omari, S.K.Q., Kilicman, A., 2012a. On generalized Hartley–Hilbert and Fourier–Hilbert transforms. Adv. Diff. Equat. 2012, 1–12. http://dx.doi.org/10.1186/1687-1847-2012-23, 2012:232.

Al-Omari, S.K.Q., Kilicman, A., 2012b. On generalized Hartley–Hilbert and Fourier–Hilbert transforms. Adv. Differ. Equat. 2012, 1–12, 2012:232.

Al-Omari, S.K.Q., Kilicman, A., 2013. Some remarks on the extended Hartley–Hilbert and Fourier–Hilbert transforms of Boehmians. Abstr. Appl. Anal. 2013, 1–6. Article ID 348701.

Beardsley, J., Mikusinski, P., 2013. A sheaf of Boehmians. Ann. Polon. Math. 107, 293–307.

Bhuvaneswari, R., Karunakaran, V., 2010. Boehmians of type S and their Fourier transforms. Ann. UMCS, Math. 64, 27–43.

Boehme, T.K., 1973. The support of Mikusinski operators. Trans. Amer. Math. Soc. 176, 319–334.

Dernek, N., Kurt, V., Simsek, Y., Yurekli, O., 2011. A generalization of the Widder potential transform and applications. Integral Transforms Spec. Funct. (6), 391–401.

Ganesan, C., 2010. Weighted ultra distributions and Boehmians. Int. J. Math. Anal. 4 (15), 703–712.

Karunakaran, V., Angeline Chella, R., 2011. Gelfand transform for a Boehmian space of analytic functions. Ann. Polon. Math. 101, 39–45.

Karunakaran, V., Ganesan, C., 2009. Fourier transform on integrable Boehmians. Integral Transforms Spec. Funct. 20, 937–941.

Loonker, D., Banerji, P.K., 2010. Wavelet transform of fractional integrals for integrable Boehmians. Applic. Appl. Math. 5 (1), 1–10.

Loonker, D., Banerji, P.K., Debnath, L., 2010. On the Hankel transform for Boehmians. Integral Transforms Spec. Funct. 21 (7), 479–486.

Mikusinski, P., 1983. Convergence of Boehmians. Jpn. J. Math. 9, 159–179.

Mikusinski, P., 1987. Fourier transform for integrable Boehmians. Rocky Mount. J. Math. 17 (3), 577–582.

Mikusinski, P., 1995. Tempered Boehmians and ultradistributions. Proc. Amer. Math. Soc. 123 (3), 813–817.

Nemzer, D., 2007. One-parameter groups of Boehmians. Bull. Korean Math. Soc. 44, 419–428.

Nemzer, D., 2010. S-asymptotic properties of Boehmians. Integral Trasforms Spec. Funct. 21 (7), 503–513.

Roopkumar, R., 2009. An extension of distributional wavelet transform. Colloq. Mathemat. 115, 195–206.

Srivastava, H.M., Singh, S.P., 1985. A note on the Widder transform related to the Poison integral for a half-plane. Int. J. Math. Edu. Sci. Technol. 16, 675–677.

Srivastava, H.M., Yürekli, O., 1991. A theorem on Widder transform and its applications. J. Math. Anal. Applic. 154, 585–593.

Widder, D.V., 1966. A transform related to the Poison integral for a half-plane. Duke Math. J. 33, 355–362.

Widder, D.V., 1971. An introduction to transform theory. Academic Press, New York.

Yürekli, O., Sadek, I., 1991. A Parseval–Goldstein theorem on the Widder potential transform and its applications, Inter. J. Math. Math. Sci. 14 (3), 517–524.

Zemanian, A.H., 1987. Generalized integral transformation. Dover Publications, Inc., New York, First published by Interscience Publishers, New York.