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تعيمم تحويل تكاملي في فضاء بوهمينز (Boehmians)

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الملخص:

هذا البحث يحقق في تحويل L^2 على بعض فضاء الدوال المعممة وقد تم بناء فضائين بوهمينز. ان تحويل L^2 تم مده وبعض خصائصه تم الحصول عليها.

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ORIGINAL ARTICLE

An extension of certain integral transform to a space of Boehmians

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Abstract This paper investigates the $\mathcal{L}^2$ transform on a certain space of generalized functions. Two spaces of Boehmians have been constructed. The transform $\mathcal{L}^2$ is extended and some of its properties are also obtained.

1. Introduction

Integral transforms are widely used to solve various problems in calculus, mechanics, mathematical physics, and some problems appear in computational mathematics as well.

In the sequence of these integral transforms, the Laplace - type integral transform, so-called $\mathcal{L}^2$ transform, is defined for a squared and an exponential function $f(t)$ by David et al. (2007), as

$$\mathcal{L}^2(f(x))(y) = \int_0^\infty xe^{-x^2y^2}f(x)dx. \tag{1}$$

The $\mathcal{L}^2$ transform is related to the classical Laplace transform by means of the following relationships :

$$\mathcal{L}^2(f(x))(y) = \frac{1}{2}\mathcal{L}\left(f\left(\sqrt{x}\right)\right)\left(y^2\right) \tag{2}$$

and

$$\mathcal{L}(f(x))(y) = 2\mathcal{L}^2\left(f\left(x^2\right)\right)\left(\sqrt{y}\right). \tag{3}$$

Let f and g be Lebesgue integrable functions; then the operation $*$ between f and g is defined by

$$(f*g)(t) = \int_0^t xf\left(\sqrt{(t^2-x^2)}\right)g(x)dx.$$

The operation $*$ is commutative, associative and satisfies the equation $\mathcal{L}^2(f*g)(y) = \mathcal{L}^2(f)(y)\mathcal{L}^2(g)(y)$.

Some facts about the transform $\mathcal{L}^2$ are given as follows:

(1) $\mathcal{L}^2\left(\frac{\sin t^2}{t^2}\right)(y) = \frac{1}{2}\arctan\left(\frac{1}{y^2}\right)$.
(2) $\mathcal{L}^2(\mathcal{H}(t-a))(y) = \frac{1}{2y^2}e^{-y^2a^2}$, $\mathcal{H}$ being the heaviside unit function.
(3) $\mathcal{L}^2(t^2)(y) = \frac{\gamma\left(\frac{n}{2}+1\right)}{2y^{n+2}}$, γ being the gamma function.

More properties, applications and the inversion formula of $\mathcal{L}^2$ transform are given by Yürekli (1999a,b).

2. Abstract construction of Boehmians

The minimal structure necessary for the construction of Boehmians consists of the following elements :

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(*i*) A set $\Im$;
(*ii*) A commutative semigroup $(\Re, *)$;
(*iii*) An operation $\odot : \Im \times \Re \to \Im$ such that for each $x \in \Im$ and $\upsilon_1, \upsilon_2, \in \Re$,

$$x \odot (\upsilon_1 * \upsilon_2) = (x \odot \upsilon_1) \odot \upsilon_2;$$

(*vi*) A collection $\Delta \subset \Re^{\mathbb{N}}$ satisfying :
  (*a*) If $x, y \in \Im, (\upsilon_n) \in \Delta, x \odot \upsilon_n = y \odot \upsilon_n$ for all n, then $x = y$;
  (*b*) If $(\upsilon_n), (\sigma_n) \in \Delta$, then $(\upsilon_n * \sigma_n) \in \Delta$. Δ is the set of all delta sequences.

Consider

$$\mathcal{A} = \{(x_n, \upsilon_n) : x_n \in \Im,\ (\upsilon_n) \in \Delta,\ x_n \odot \upsilon_m = x_m \odot \upsilon_n,\ \forall m,\ n \in \mathbb{N}\}.$$

If $(x_n, \upsilon_n), (y_n, \sigma_n) \in \mathcal{A}, x_n \odot \sigma_m = y_m \odot \upsilon_n, \forall m, n \in \mathbb{N}$, then we say $(x_n, \upsilon_n) \sim (y_n, \sigma_n)$. The relation $\sim$ is an equivalence relation in $\mathcal{A}$. The space of equivalence classes in $\mathcal{A}$ is denoted by $\boldsymbol{b}(\Im, \Re, \Delta)$. Elements of $\boldsymbol{b}(\Im, \Re, \Delta)$ are called Boehmians.

Between $\Im$ and $\boldsymbol{b}(\Im, \Re, \Delta)$ there is a canonical embedding expressed as

$$x \to \frac{x \odot s_n}{s_n} \text{ as } n \to \infty.$$

The operation $\odot$ can be extended to $\boldsymbol{b}(\Im, \Re, \Delta) \odot \Im$ by

$$\frac{x_n}{\upsilon_n} \odot t = \frac{x_n \odot t}{\upsilon_n}.$$

The sum of two Boehmians and multiplication by a scalar can be defined in a natural way

$$\left[\frac{(x_n)}{(\upsilon_n)}\right] + \left[\frac{(g_n)}{(\psi_n)}\right] = \left[\frac{(x_n \odot \psi_n + g_n \odot \upsilon_n)}{(\upsilon_n \odot \psi_n)}\right]$$

and

$$\alpha\left[\frac{(x_n)}{(\upsilon_n)}\right] = \left[\frac{\alpha x_n}{\upsilon_n}\right],\ \alpha \in \mathbb{C}.$$

The operation $\odot$ and the differentiation are defined by

$$\left[\frac{(x_n)}{(\upsilon_n)}\right] \odot \left[\frac{(g_n)}{(\psi_n)}\right] = \left[\frac{(x_n \odot g_n)}{(\upsilon_n \odot \psi_n)}\right]$$

and

$$D^\alpha\left[\frac{(x_n)}{(\upsilon_n)}\right] = \left[\frac{(D^\alpha x_n)}{(\upsilon_n)}\right].$$

In particular, if $\left[\frac{(x_n)}{(\upsilon_n)}\right] \in \boldsymbol{b}(\Im, \Re, \Delta)$ and $\delta \in \Re$ is any fixed element, then the product $\odot$, defined by

$$\left[\frac{(x_n)}{(\upsilon_n)}\right] \odot \delta = \left[\frac{(x_n \odot \delta)}{(\upsilon_n)}\right],$$

is in $\boldsymbol{b}(\Im, \Re, \Delta)$.

Many a time $\Im$(also considered as a quasi-normed space) is also equipped with a notion of convergence. The intrinsic relationship between the notion of convergence and the product $\odot$ is given by:

(*i*) If $f_n \to f$ as $n \to \infty$ in $\Im$ and $\phi \in \Re$ is any fixed element, then $f_n \odot \phi \to f \odot \phi$ as $n \to \infty$ in $\Im$.
(*ii*) If $f_n \to f$ as $n \to \infty$ in $\Im$ and $(\delta_n) \in \Delta$, then $f_n \odot \delta_n \to f$ as $n \to \infty$ in $\Im$.

In $\boldsymbol{b}(\Im, \Re, \Delta)$, two types of convergence are:

(1) A sequence (h_n) in $\boldsymbol{b}(\Im, \Re, \Delta)$ is said to be δ-convergent to h in $\boldsymbol{b}(\Im, \Re, \Delta)$, denoted by $h_n \xrightarrow{\delta} h$ as $n \to \infty$, if there exists a delta sequence (υ_n) such that $(h_n \odot \upsilon_n), (h \odot \upsilon_n) \in \Im, \forall k$, $n \in \mathbb{N}$, and $(h_n \odot \upsilon_k) \to (h \odot \upsilon_k)$ as $n \to \infty$, in $\Im$, for every $k \in \mathbb{N}$.
(2) A sequence (h_n) in $\boldsymbol{b}(\Im, \Re, \Delta)$ is said to be Δ- convergent to h in $\boldsymbol{b}(\Im, \Re, \Delta)$, denoted by $h_n \xrightarrow{\Delta} h$ as $n \to \infty$, if there exists a $(\upsilon_n) \in \Delta$ such that $(h_n - h) \odot \upsilon_n \in \Im$, $\forall n \in \mathbb{N}$, and $(h_n - h) \odot \upsilon_n \to 0$ as $n \to \infty$ in $\Im$.

The following theorem is equivalent to the statement of δ- convergence :

Preposition 1. *$h_n \xrightarrow{\delta} h(n \to \infty)$ in $\boldsymbol{b}(\Im, \Re, \Delta)$ if and only if there is $f_{n,k}$, $f_k \in \Im$ and $\upsilon_k \in \Delta$ such that $h_n = \left[\frac{f_{n,k}}{\upsilon_k}\right]$, $h = \left[\frac{f_k}{\upsilon_k}\right]$ and for each $k \in \mathbb{N}$, $f_{n,k} \to f_k$ as $n \to \infty$ in $\Im$.*

For further discussion of Boehmian spaces and their construction; see Ganesan (2010), Karunakaran and Ganesan (2009), Al-Omari (2012, 2013a,b,c,d), Al-Omari and Kilicman (2011, 2012a,b, 2013a,b), Boehme (1973), Bhuvaneswari and Karunakaran (2010), Ganesan (2010), Karunakaran and Angeline (2011), Karunakaran and Devi (2010), Mikusinski (1983, 1987, 1995), Nemzer (2006, 2007, 2008, 2009, 2010) and Roopkumar (2009).

3. The Boehmian space $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet,)$

$\boldsymbol{p}$ denotes the space of rapidly decreasing functions defined on $\mathbb{R}_+(\mathbb{R}_+ = (0, \infty))$. That is, $\phi(x) \in \boldsymbol{p}$ if $\phi(x)$ is a complex-valued and infinitely smooth function defined on $\mathbb{R}_+$ and is such that, as $|t| \to \infty$, ϕ and its partial derivatives decrease to zero faster than every power of $|t|^{-1}$.

In more details, $\phi(t) \in \boldsymbol{p}$ iff it is infinitely smooth and is such that

$$|t^m \phi^{(k)}(t)| \leqslant C_{m,k},\ \ t \in \mathbb{R}_+, \tag{4}$$

m and k run through all non-negative integers; see Pathak (1997).

$\boldsymbol{d}$ denotes the Schwartz space of test functions of bounded support defined on $\mathbb{R}_+$.

$\bullet$ denotes the Mellin-type convolution product **of**first kind defined by Zemanian (1987), as

$$(f \bullet g)(y) = \int_0^\infty f\left(\frac{y}{t}\right) t^{-1} g(t) dt. \tag{5}$$

To construct the first Boehmian space $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$, we need to establish the following necessary theorems.

Theorem 2. *Let $\phi \in \boldsymbol{p}$ and $\varphi \in \boldsymbol{d}$; then we have $\phi \bullet \varphi \in \boldsymbol{p}$.*

Proof. Let $\mathbb{K}$ be a compact subset in $\mathbb{R}_+$containing the support of φ. Then, for all $k \in \mathbb{N}$ and $m \in \mathbb{N}$, we, by (4) and (5), get that

$$\left|y^m \mathcal{D}_y^k (\phi \bullet \varphi)(y)\right| \leqslant \int_{\mathbb{K}} \left|y^m \mathcal{D}_y^k \left(\phi\left(\frac{y}{t}\right) t^{-1} \varphi(t)\right)\right| dt$$

$$= \int_{\mathbb{K}} |y^m| \left|\mathcal{D}_y^k \phi\left(\frac{y}{t}\right)\right| |t^{-1}\varphi(t)| dt \leqslant C_{m,k} \int_{\mathbb{K}} |t^{-1}\varphi(t)| dt < MC_{m,k}.$$

Hence, by considering the supremum over all $y \in \mathbb{R}_+$, we obtain that $\phi \bullet \varphi \in \boldsymbol{p}$.

This completes the proof of the theorem. □

Theorem 3. *Let $\phi_1, \phi_2 \in \boldsymbol{p}, \varphi_1, \varphi_2 \in \boldsymbol{d}$; then we have*

(1) $\phi_1 \bullet \phi_2 = \phi_2 \bullet \phi_1$.
(2) $(\phi_1 + \phi_2) \bullet \varphi_1 = \phi_1 \bullet \varphi_1 + \phi_2 \bullet \varphi_1$.
(3) $(\phi_1 \bullet \varphi_1) \bullet \varphi_2 = \phi_1 \bullet (\varphi_1 \bullet \varphi_2)$.

Proof of (1) and (3) follows from the Reference (Pathak, 1997). Proof of (2) is straightforward from the properties of integration. Therefore, we prefer to omit the details.

Hence the theorem is completely proved.

Theorem 4. *Let $\phi_n \to \phi$ in $\boldsymbol{p}$ as $n \to \infty$ and $\varphi \in \boldsymbol{d}$; then $\phi_n \bullet \varphi \to \phi \bullet \varphi$ as $n \to \infty$.*

Proof of this theorem is straightforward from simple integration.

Definition 5. Denoted by Δ the set of all sequences $(\delta_n) \in \boldsymbol{d}$ that satisfying:

(*i*) $\int_0^\infty \delta_n = 1$;
(*ii*) $\int_0^\infty |\delta_n| < M,\ n \in \mathbb{N}$;
(*iii*) supp $\delta_n \subset (0, \varepsilon),\ \varepsilon \to 0$ as $n \to \infty$.

Each (δ_n) is called a delta sequence or an approximating identity which corresponds to the Dirac delta distribution.

Theorem 6. *Let $\phi \in \boldsymbol{p}$ and $(\delta_n) \in \Delta$; then $\phi \bullet \delta_n \to \phi$ as $n \to \infty$.*

Proof. By Axiom (*i*) of Definition 5, we write

$$|(\phi \bullet \delta_n - \phi)(y)| \leqslant \int_0^\infty \left|\phi\left(\frac{y}{t}\right)t^{-1} - \phi(y)\right||\delta_n(t)|dt. \quad (6)$$

The mapping $\psi(y) = \phi\left(\frac{y}{t}\right)t^{-1} - \phi(y)$ is uniformly continuous for each $y \in \mathbb{R}_+$. Therefore, it follows that

$$|(\phi \bullet \delta_n - \phi)(y)| \to 0$$

as $n \to \infty$.

Hence, we have established that $\phi \bullet \delta_n \to \phi$ as $n \to \infty$.

Hence the theorem has been proved. □

Now, we assert that the product of delta sequences is a delta sequence. Detailed proof is as follows.

Let $\phi_n, \psi_n \in \Delta$. Then, we have

$$\int_0^\infty (\phi_n \bullet \psi_n)(x)dx = \int_0^\infty \int_0^\infty \phi_n(x/y)\psi_n(y)y^{-1}\ dydx$$

$$= \int_0^\infty \int_0^\infty \phi_n(x/y)dx\psi_n(y)y^{-1}dy \quad \text{(By change of variables)}$$

$$= \int_0^\infty \int_0^\infty \phi_n(u)du\psi_n(y)dy = \int_0^\infty \phi_n(u)du \int_0^\infty \psi_n(y)dy = 1$$

Next, let M_1 *and* M_2 be constants such that $\int_0^\infty |\phi_n| \leqslant M_1$ and $\int_0^\infty |\psi_n| \leqslant M_2$; then

$$\int_0^\infty |(\phi_n \bullet \psi_n)(x)|dx \leqslant \int_0^\infty \int_0^\infty |\phi_n(x/y)\phi_n(y)|y^{-1}dydx.$$

Once again, change of variables, implies

$$\int_0^\infty |(\phi_n \triangleright \psi_n)(x)|dx \leqslant \int_0^\infty |\phi_n(u)|du \int_0^\infty |\psi_n(y)|dy \leqslant M_1M_2.$$

Finally, the inequality

$$\text{supp}(\phi_n \bullet \psi_n) \leqslant (\text{supp } \phi_n)(\text{supp } \psi_n).$$

establishes our assertion.

The space $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$ is therefore considered as a space of Boehmians.

Definition 7. Let $\phi \in \boldsymbol{p}$ and $\varphi \in \boldsymbol{d}$. Between $\boldsymbol{p}$ and $\boldsymbol{d}$ define a product $\times$ as in the integral equation

$$(\phi \times \varphi)(y) = \int_0^\infty t\phi(yt)\varphi(t)dt. \quad (7)$$

We establish the second Boehmian space $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \times, \Delta)$.

Theorem 8. *Let $\phi \in \boldsymbol{p}$ and $\varphi \in \boldsymbol{d}$; then we have $\phi \times \varphi \in \boldsymbol{p}$.*

Proof. By (7), we write

$$\left|y^m\mathcal{D}_y^k(\phi \times \varphi)(y)\right| = \left|y^m\mathcal{D}_y^k \int_0^\infty t\phi(yt)\varphi(t)dt\right|$$
$$\leqslant \int_0^\infty \left|y^m\mathcal{D}_y^k\phi(yt)\right||t\varphi(t)|dt. \quad (8)$$

Since $\phi \in \boldsymbol{p}$, we, from (8), find that

$$\left|y^m\mathcal{D}_y^k(\phi \times \varphi)(y)\right| \leqslant C_{m,k}\int_{\mathbb{K}} |t\varphi(t)|dt < MC_{m,k}, \quad (9)$$

where $\mathbb{K}$ is a compact set in $\mathbb{R}_+$ containing the support of φ.

Hence, (9) leads to the conclusion that $\phi \times \varphi \in \boldsymbol{p}$.

This completes the proof of the theorem. □

Theorem 9. *Let $\phi \in \boldsymbol{p}$ and $\varphi, \psi \in \boldsymbol{d}$; then $\phi \times (\varphi \bullet \psi) = (\phi \times \varphi) \times \psi$.*

Proof. Let the hypothesis of the theorem satisfies for $\phi \in \boldsymbol{p}$ and $\varphi, \psi \in \boldsymbol{d}$. Then, using (5) and (7) yield

$$(\phi \times (\varphi \bullet \psi))(y) = \int_0^\infty \phi(yt)t(\varphi \bullet \psi)(t)dt$$
$$= \int_0^\infty \phi(yt)t \int_0^\infty \varphi\left(\frac{t}{\xi}\right)\xi^{-1}\psi(\xi)d\xi dt$$
$$= \int_0^\infty \left(\int_0^\infty \phi(yt)t\varphi\left(\frac{t}{\xi}\right)dt\right)\psi(\xi)\xi^{-1}d\xi. \quad (10)$$

The change of variables $t = \xi z$ implies $dt = \xi dz$.

Hence, from (10), we obtain

$$(\phi \times (\varphi \bullet \psi))(y) = \int_0^\infty \left(\int_0^\infty \phi(y\xi z)\xi z\varphi(z)\xi dz\right)\psi(\xi)\xi^{-1}d\xi$$
$$= \int_0^\infty \left(\int_0^\infty \phi((y\xi)z)z\varphi(z)dz\right)\xi\psi(\xi)d\xi$$
$$= \int_0^\infty (\phi \times \varphi)(y\xi)\xi\psi(\xi)d\xi$$
$$= ((\phi \times \varphi) \times \psi)(y).$$

This completes the proof of the theorem. □

Theorem 10. *Let $\phi_1, \phi_2 \in \boldsymbol{p}$ and $\varphi \in \boldsymbol{d}$; then we have*

(*i*) $(\phi_1 + \phi_2) \times \varphi = \phi_1 \times \varphi + \phi_2 \times \varphi$.
(*ii*) $(\alpha\phi) \times \varphi = \alpha(\phi \times \varphi),\ \alpha \in \mathbb{C}$.

Proof of this theorem follows from the general properties of integration. Details are thus omitted. The theorem is therefore proved.

Theorem 11. *Let $\phi_n \to \phi$ in $\boldsymbol{p}$ as $n \to \infty$; then $\phi_n \times \varphi \to \phi \times \varphi$ as $n \to \infty$, for each $\varphi \in \boldsymbol{d}$.*

Proof of this theorem follows from Theorem 9.

Theorem 12. *Let $\phi \in \boldsymbol{p}$ and $(\delta_n) \in \Delta$; then $\phi \times \delta_n \to \phi$ as $n \to \infty$.*

Proof. Utilizing Definition 5 and Axiom (*i*) yield

$$\begin{aligned} &\left| y^m \mathcal{D}_y^k((\phi \times \delta_n)(y) - \phi(y)) \right| \\ &= \left| y^m \int_0^\infty \mathcal{D}_y^k(t\phi(yt))\delta_n(t)dt - y^m \mathcal{D}_y^k \phi(y) \int_0^\infty \delta_n(t)dt \right| \\ &\leqslant \left| y^m \mathcal{D}_y^k(t\phi(yt) - \phi(y)) \right| |\delta_n(t)| dt \end{aligned} \tag{11}$$

Since $\phi \in \boldsymbol{p}$ it follows

$$\left| y^m \mathcal{D}_y^k(t\phi(yt) - \phi(y)) \right| < C_{m,k},\ m,\ k \in \mathbb{N}.$$

Hence, from(11) and Definition 5(*ii*), we get

$$\left| y^m \mathcal{D}_y^k((\phi \times \delta_n)(y) - \phi(y)) \right| < MC_{m,k}.$$

Hence the theorem is therefore completely proved. □

The Boehmian space $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \times, \Delta)$ has been constructed.

Theorem 13. *Let $\phi \in \boldsymbol{p}$ and $\varphi \in \boldsymbol{d}$; then we have*

$$\mathcal{L}^2(\phi \bullet \varphi)(y) = (\mathcal{L}^2\phi \times \varphi)(y).$$

Proof. Let $\phi \in \boldsymbol{p}$ and $\varphi \in \boldsymbol{d}$; then we have

$$\begin{aligned} \mathcal{L}^2(\phi \bullet \varphi)(y) &= \int_0^\infty (\phi \bullet \varphi)(\xi)\xi e^{-y^2\xi^2} d\xi \\ &= \int_0^\infty \left(\int_0^\infty \phi\left(\frac{\xi}{t}\right) t^{-1}\varphi(t)dt \right) \xi e^{-y^2\xi^2} d\xi \\ &= \int_0^\infty \left(\int_0^\infty \phi\left(\frac{\xi}{t}\right) \xi e^{-y^2\xi^2} d\xi \right) t^{-1}\varphi(t)dt. \end{aligned}$$

The substitution $\xi = zt$ implies

$$\begin{aligned} \mathcal{L}^2(\phi \bullet \varphi)(y) &= \int_0^\infty \left(\int_0^\infty \phi(z) zte^{-y^2z^2t^2} tdz \right) t^{-1}\varphi(t)dt \\ &= \int_0^\infty \left(\int_0^\infty \phi(z) ze^{-(y^2t^2)z^2} dz \right) t\varphi(t)dt \\ &= \int_0^\infty \mathcal{L}^2(\phi)(yt)t\varphi(t)dt \\ &= (\mathcal{L}^2\phi \times \varphi)(y). \end{aligned}$$

Hence the theorem is proved. □

In view of above, we define the extended $\mathcal{L}^2$ transform of a Boehmian $\left[\frac{(\phi_n)}{(\delta_n)}\right]$ in $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$ as

$$\widehat{\mathcal{L}^2}\left[\frac{(\phi_n)}{(\delta_n)}\right] = \left[\frac{(\mathcal{L}^2\phi_n)}{(\delta_n)}\right] \in \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \times, \Delta). \tag{12}$$

Theorem 14. $\widehat{\mathcal{L}^2} : \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta) \to \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \times, \Delta)$ *is well-defined.*

Proof. Let $\left[\frac{(u_n)}{(r_n)}\right] = \left[\frac{(v_n)}{(\psi_n)}\right]$ in $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$. Then, by the concept of quotient of sequences in $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$, we write

$u_n \bullet \psi_m = v_m \bullet r_n = v_n \bullet r_m.$

Employing $\mathcal{L}^2$ on both sides of the above equation implies

$\mathcal{L}^2 u_n \times \psi_m = \mathcal{L}^2 v_n \times r_m,\ n,\ m.$

That is,

$\frac{\mathcal{L}^2 u_n}{r_n}$ is equivalent to $\frac{\mathcal{L}^2 v_n}{\psi_n}$.

Therefore, we get

$$\left[\frac{(\mathcal{L}^2 u_n)}{(r_n)}\right] = \left[\frac{(\mathcal{L}^2 v_n)}{(\psi_n)}\right].$$

This completes the proof of the theorem. □

It is also interesting to know that the transform $\widehat{\mathcal{L}^2}$ is a linear mapping from $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$ into $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \times, \Delta)$.

Detailed proof can be given as follows : If $\left[\frac{(u_n)}{(r_n)}\right]$, $\left[\frac{(v_n)}{(\psi_n)}\right]$ $\in \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$ then

$$\begin{aligned} \widehat{\mathcal{L}^2}\left(\left[\frac{(u_n)}{(r_n)}\right] + \left[\frac{(v_n)}{(\psi_n)}\right]\right) &= \widehat{\mathcal{L}^2}\left(\left[\frac{(u_n \bullet \psi_n) + (v_n \bullet r_n)}{(r_n \bullet \psi_n)}\right]\right) \\ &= \left[\frac{\mathcal{L}^2(u_n \bullet \psi_n + v_n \bullet r_n)}{(r_n \times \psi_n)}\right] \\ &= \left[\frac{\mathcal{L}^2(u_n \bullet \psi_n) + \mathcal{L}^2(v_n \bullet r_n)}{(r_n \times \psi_n)}\right] \\ &= \left[\frac{(\mathcal{L}^2 u_n \times \psi_n) + (\mathcal{L}^2 v_n \times r_n)}{(r_n \times \psi_n)}\right] \\ &= \left[\frac{(\mathcal{L}^2 u_n)}{(r_n)}\right] + \left[\frac{(\mathcal{L}^2 v_n)}{(\psi_n)}\right]. \end{aligned}$$

Hence,

$$\widehat{\mathcal{L}^2}\left(\left[\frac{(u_n)}{(r_n)}\right] + \left[\frac{(v_n)}{(\psi_n)}\right]\right) = \widehat{\mathcal{L}^2}\left[\frac{(u_n)}{(r_n)}\right] + \widehat{\mathcal{L}^2}\left[\frac{(v_n)}{(\psi_n)}\right].$$

Also, if $\alpha \in \mathbb{C}$, the field of complex numbers, then we see that

$$\alpha\widehat{\mathcal{L}^2}\left[\frac{(u_n)}{(r_n)}\right] = \alpha\left[\frac{(\mathcal{L}^2 u_n)}{(r_n)}\right] = \left[\frac{(\mathcal{L}^2(\alpha u_n))}{(r_n)}\right].$$

Hence,

$$\alpha\widehat{\mathcal{L}^2}\left[\frac{(u_n)}{(r_n)}\right] = \widehat{\mathcal{L}^2}\left(\alpha\left[\frac{(u_n)}{(r_n)}\right]\right).$$

This completes the proof of the theorem.

Definition 15. Let $\left[\frac{(\mathcal{L}^2 f_n)}{(\phi_n)}\right] \in \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \times, \Delta)$; then we define the inverse $\widehat{\mathcal{L}^2}$ transform of $\left[\frac{(\mathcal{L}^2 f_n)}{(\phi_n)}\right]$ as

$$\widehat{\mathcal{L}^2}^{-1}\left[\frac{(\mathcal{L}^2 f_n)}{(\phi_n)}\right] = \left[\frac{(\mathcal{L}^2)^{-1}(\mathcal{L}^2 f_n)}{(\phi_n)}\right] = \left[\frac{(f_n)}{(\phi_n)}\right], \tag{13}$$

for each $(\phi_n) \in \Delta$.

Theorem 16. $\widehat{\mathcal{L}^2} : \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta) \to \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \times, \Delta)$ *is an isomorphism.*

Proof. Assume $\widehat{\mathcal{L}^2}\left[\frac{(f_n)}{(\phi_n)}\right] = \widehat{\mathcal{L}^2}\left[\frac{(g_n)}{(\psi_n)}\right]$. Using (12) and the concept of quotients we get $\mathcal{L}^2 f_n \times \psi_m = \mathcal{L}^2 g_m \times \phi_n$. Therefore, Theorem 13 implies

$$\mathcal{L}^2(f_n \bullet \psi_m) = \mathcal{L}^2(g_m \bullet \phi_n).$$

Properties of $\mathcal{L}^2$ implies $f_n \bullet \psi_m = g_m \bullet \phi_n$. Therefore, from the concept of qoutients of equivalent classes of $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$, we get

$$\left[\frac{(f_n)}{(\phi_n)}\right] = \left[\frac{(g_n)}{(\psi_n)}\right].$$

To establish that $\widehat{\mathcal{L}^2}$ is surjective, let $\left[\frac{(\mathcal{L}^2 f_n)}{(\phi_n)}\right] \in \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \times, \Delta)$. Then $\mathcal{L}^2 f_n \times \phi_m = \mathcal{L}^2 f_m \times \phi_n$ for every $m, n \in \mathbb{N}$. Once again, Theorem 13 implies $\mathcal{L}^2(f_n \bullet \phi_m) = \mathcal{L}^2(f_m \bullet \phi_n)$. Hence $\left[\frac{(f_n)}{(\phi_n)}\right] \in \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$ is a Boehmian satisfying the equation

$$\widehat{\mathcal{L}^2}\left[\frac{(f_n)}{(\phi_n)}\right] = \left[\frac{(\mathcal{L}^2 f_n)}{(\phi_n)}\right].$$

This completes the proof of the theorem. □

Theorem 17. *Let* $\left[\frac{(\mathcal{L}^2 f_n)}{(\phi_n)}\right] \in \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \times, \Delta)$ *and* $\phi \in \boldsymbol{d}$; *then we have*

$$\widehat{\mathcal{L}^2}^{-1}\left(\left[\frac{(\mathcal{L}^2 f_n)}{(\phi_n)}\right] \times \phi\right) = \left[\frac{(f_n)}{(\phi_n)}\right] \bullet \phi \text{ and}$$
$$\widehat{\mathcal{L}^2}\left(\left[\frac{(f_n)}{(\phi_n)}\right] \bullet \phi\right) = \left[\frac{(\mathcal{L}^2 f_n)}{(\phi_n)}\right] \times \phi.$$

Detailed proof of the first part is as follows : Applying (13) yields

$$\widehat{\mathcal{L}^2}^{-1}\left(\left[\frac{(\mathcal{L}^2 f_n)}{(\phi_n)}\right] \times \phi\right) = \widehat{\mathcal{L}^2}^{-1}\left(\left[\frac{(\mathcal{L}^2 f_n) \times \phi}{(\phi_n)}\right]\right) = \left[\frac{(\mathcal{L}^2)^{-1}((\mathcal{L}^2 f_n) \times \phi)}{(\phi_n)}\right].$$

By using (12), we obtain

$$\widehat{\mathcal{L}^2}^{-1}\left(\left[\frac{(\mathcal{L}^2 f_n)}{(\phi_n)}\right] \times \phi\right) = \left[\frac{(f_n) \bullet \phi}{(\phi_n)}\right] = \left[\frac{(f_n)}{(\phi_n)}\right] \bullet \phi.$$

The proof of the second part $\widehat{\mathcal{L}^2}\left(\left[\frac{(f_n)}{(\phi_n)}\right] \bullet \phi\right) = \left[\frac{(\mathcal{L}^2 f_n)}{(\phi_n)}\right] \times \phi$ is similar.

This completes the proof of the theorem.

Theorem 18. $\widehat{\mathcal{L}^2} : \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta) \to \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \times, \Delta)$*and* $\widehat{\mathcal{L}^2}^{-1} : \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \times, \Delta) \to \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$ *are continuous with respect to* δ *and* Δ*-convergence.*

Proof. First of all, we show that $\widehat{\mathcal{L}^2} : \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta) \to \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \times, \Delta)$ and $\widehat{\mathcal{L}^2}^{-1} : \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \times, \Delta) \to \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$ are continuous with respect to δ-convergence.

Let $\beta_n \xrightarrow{\delta} \beta$ in $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$ as $n \to \infty$; then we show that $\widehat{\mathcal{L}^2}\beta_n \to \widehat{\mathcal{L}^2}\beta$ as $n \to \infty$. By virtue of Preposition 1 we can find $f_{n,k}$ and f_k in $\boldsymbol{p}$ such that

$$\beta_n = \left[\frac{f_{n,k}}{\phi_k}\right] \text{ and } \beta = \left[\frac{f_k}{\phi_k}\right]$$

and $f_{n,k} \to f_k$ as $n \to \infty$ for every $k \in \mathbb{N}$. Hence, $\mathcal{L}^2 f_{n,k} \to \mathcal{L}^2 f_k$ as $n \to \infty$ in the space $\boldsymbol{p}$. Thus,

$$\left[\frac{\mathcal{L}^2 f_{n,k}}{\phi_k}\right] \to \left[\frac{\mathcal{L}^2 f_k}{\phi_k}\right]$$

as $n \to \infty$ in $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \times, \Delta)$.

To prove the second part, let $g_n \xrightarrow{\delta} g$ in $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \times, \Delta)$ as $n \to \infty$. Then, once again, by Preposition 1, $g_n = \left[\frac{\mathcal{L}^2 f_{n,k}}{\phi_k}\right]$ and $g = \left[\frac{\mathcal{L}^2 f_k}{\phi_k}\right]$ and $\mathcal{L}^2 f_{n,k} \to \mathcal{L}^2 f_k$ as $n \to \infty$. Hence $f_{n,k} \to f_k$ in $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$ as $n \to \infty$. Or, $\left[\frac{f_{n,k}}{\phi_k}\right] \to \left[\frac{f_k}{\phi_k}\right]$ as $n \to \infty$. Using (13) we get

$$\widehat{\mathcal{L}^2}^{-1}\left[\frac{\mathcal{L}^2 f_{n,k}}{\phi_k}\right] \to \widehat{\mathcal{L}^2}^{-1}\left[\frac{\mathcal{L}^2 f_k}{\phi_k}\right] \text{ as } n \to \infty.$$

Now, we establish continuity of $\widehat{\mathcal{L}^2}$ and $\widehat{\mathcal{L}^2}^{-1}$ with respect to Δ-convergence.

Let $\beta_n \xrightarrow{\Delta} \beta$ in $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$ as $n \to \infty$. Then, there exist $f_n \in \boldsymbol{p}$ and $(\phi_n) \in \Delta$ such that $(\beta_n - \beta) \bullet \phi_n = \left[\frac{(f_n) \bullet \phi_k}{\phi_k}\right]$ and $f_n \to 0$ as $n \to \infty$.

Employing (12) gives

$$\widehat{\mathcal{L}^2}((\beta_n - \beta) \bullet \phi_n) = \left[\frac{\mathcal{L}^2((f_n) \bullet \phi_k)}{\phi_k}\right].$$

Hence, we have

$$\widehat{\mathcal{L}^2}((\beta_n - \beta) \bullet \phi_n) = \left[\frac{(\mathcal{L}^2 f_n) \times \phi_k}{\phi_k}\right] = \mathcal{L}^2 f_n \to 0$$

as $n \to \infty$ in $\boldsymbol{p}$.

Therefore

$$\widehat{\mathcal{L}^2}((\beta_n - \beta) \bullet \phi_n) = \left(\widehat{\mathcal{L}^2}\beta_n - \widehat{\mathcal{L}^2}\beta\right) \times \phi_n \to 0 \text{ as } n \to \infty.$$

Hence, $\widehat{\mathcal{L}^2}\beta_n \xrightarrow{\Delta} \widehat{\mathcal{L}^2}\beta$ as $n \to \infty$.

Finally, let $g_n \xrightarrow{\Delta} g$ in $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \times, \Delta)$ as $n \to \infty$; then we find $\mathcal{L}^2 f_k \in \boldsymbol{p}$ such that $(g_n - g) \times \phi_n = \left[\frac{\mathcal{L}^2 f_k \times \phi_k}{\phi_k}\right]$ and $\mathcal{L}^2 f_k \to 0$ as $n \to \infty$for some $(\phi_n) \in \Delta$.

Now, using (13), we obtain

$$\widehat{\mathcal{L}^2}^{-1}((g_n - g) \times \phi_n) = \left[\frac{(\mathcal{L}^2)^{-1}(\mathcal{L}^2 f_k \times \phi_k)}{\phi_k}\right].$$

Theorem 13 implies

$$\widehat{\mathcal{L}^2}^{-1}((g_n - g) \times \phi_n) = \left[\frac{(f_n) \bullet \phi_k}{\phi_k}\right] = f_n \to 0 \text{ as } n \to \infty \text{ in } \boldsymbol{p}.$$

Thus

$$\widehat{\mathcal{L}^2}^{-1}((g_n - g) \times \phi_n) = \left(\widehat{\mathcal{L}^2}^{-1} g_n - \widehat{\mathcal{L}^2}^{-1} g\right) \bullet \phi_n \to 0 \text{ as } n \to \infty.$$

From this we find that $\widehat{\mathcal{L}^2}^{-1} g_n \xrightarrow{\Delta} \widehat{\mathcal{L}^2}^{-1} g$ as $n \to \infty$ in $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$.

This completes the proof of the theorem. □

Theorem 19. *The extended $\widehat{\mathcal{L}^2}$ transform is consistent with $\mathcal{L}^2$.*

Proof. For every $f \in \boldsymbol{p}$, let β be its representative in $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$; then $\beta = \left[\frac{f \bullet (\varphi_n)}{(\varphi_n)}\right]$, where $(\varphi_n) \in \Delta$, $\forall n \in \mathbb{N}$. Its clear that (φ_n) is independent from the representative, $\forall n \in \mathbb{N}$. Therefore

$$\widehat{\mathcal{L}^2}(\beta) = \widehat{\mathcal{L}^2}\left(\left[\frac{f \bullet (\varphi_n)}{(\varphi_n)}\right]\right)$$

$$\text{i.e.} = \left[\frac{\mathcal{L}^2(f \bullet (\varphi_n))}{(\varphi_n)}\right]$$

$$= \left[\frac{\mathcal{L}^2 f \times (\varphi_n)}{(\varphi_n)}\right]$$

which is the representative of $\mathcal{L}^2 f$ in $\boldsymbol{p}$.

Hence the proof is completed. □

Theorem 20. *Let $\left[\frac{(g_n)}{(\psi_n)}\right] \in \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \times, \Delta)$; then the necessary and sufficient condition that $\left[\frac{(g_n)}{(\psi_n)}\right]$ to be in the range of $\widehat{\mathcal{L}^2}$ is that g_n belongs to range of $\mathcal{L}^2$ for every $n \in \mathbb{N}$.*

Proof. Let $\left[\frac{(g_n)}{(\psi_n)}\right]$ be in the range of $\widehat{\mathcal{L}^2}$; then of course g_n belongs to the range of $\mathcal{L}^2, \forall n \in \mathbb{N}$.

To establish the converse, let g_n be in the range of $\mathcal{L}^2$, $\forall n \in \mathbb{N}$. Then there is $f_n \in \boldsymbol{p}$ such that $\mathcal{L}^2 f_n = g_n$, $n \in \mathbb{N}$.

Since, $\left[\frac{(g_n)}{(\psi_n)}\right] \in \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \times, \Delta)$,

$$g_n \times \psi_m = g_m \times \psi_n,$$

$\forall m, n \in \mathbb{N}$. Therefore,

$$\mathcal{L}^2(f_n \bullet \varphi_n) = \mathcal{L}^2(f_m \bullet \varphi_n), \ \forall m, n \in \mathbb{N},$$

where $f_n \in \boldsymbol{p}$ and $\varphi_n \in \Delta$, $\forall n \in \mathbb{N}$.

The fact that $\mathcal{L}^2$ is injective, implies that $f_n \bullet \varphi_m = f_m \bullet \varphi_n$, $m, n \in \mathbb{N}$.

Thus, $\frac{f_n}{\varphi_n}$ is qoutient of sequences in $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$. Hence,

$$\left[\frac{(f_n)}{(\varphi_n)}\right] \in \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta) \text{ and } \widehat{\mathcal{L}^2}\left(\left[\frac{(f_n)}{(\varphi_n)}\right]\right) = \left[\frac{(g_n)}{(\psi_n)}\right].$$

Hence the theorem is proved. □

Theorem 21. *Let $\beta = \left[\frac{(f_n)}{(\varphi_n)}\right] \in \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$ and $\gamma = \left[\frac{(\kappa_n)}{(\phi_n)}\right] \in \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$; then*

$$\widehat{\mathcal{L}^2}(\beta \bullet \gamma) = \widehat{\mathcal{L}^2}\beta \times \gamma.$$

Proof. Assume the requirements of the theorem are satisfied for some β and $\gamma \in \boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$. Then

$$\widehat{\mathcal{L}^2}(\beta \bullet \gamma) = \widehat{\mathcal{L}^2}\left(\left[\frac{(f_n) \bullet (\kappa_n)}{(\varphi_n) \bullet (\phi_n)}\right]\right)$$

$$= \left[\frac{\mathcal{L}^2((f_n) \bullet (\kappa_n))}{(\varphi_n) \bullet (\phi_n)}\right]$$

$$= \left[\frac{(\mathcal{L}^2 f_n) \times (\kappa_n)}{(\varphi_n) \times (\phi_n)}\right]$$

$$= \left[\frac{(\mathcal{L}^2 f_n)}{(\varphi_n)}\right] \times \left[\frac{(\kappa_n)}{(\phi_n)}\right].$$

Therefore,

$$\widehat{\mathcal{L}^2}(\beta \bullet \gamma) = \widehat{\mathcal{L}^2}(\beta) \times \gamma.$$

This completes the proof of the theorem. □

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