Journal of the Association of Arab Universities for Basic and Applied Sciences (2015) **17**, 14–19

University of Bahrain

Journal of the Association of Arab Universities for Basic and Applied Sciences

www.elsevier.com/locate/jaaubas
www.sciencedirect.com

نموذج كسري لمعادلة نافيير- ستوكس لمائع لزج ذات سريان غير مستقر

Devendra Kumar [a,*], Jagdev Singh [b], Sunil Kumar [c]

[a] Department of Mathematics, Jagan Nath Gupta Institute of Engineering and Technology, Jaipur 302022, Rajasthan, India.
[b] Department of Mathematics, Jagan Nath University, Village-Rampura, Tehsil-Chaksu, Jaipur 303901, Rajasthan, India.
[c] Department of Mathematics, National Institute of Technology, Jamshedpur 831014, Jharkhand, India.

الملخص:

في هذا البحث تم تقديم أسلوب عملي على أساس طريقة التحويل الحديثة لاضطراب هوموتوبي (HPTM) لحل معادلة نافيير–ستوكس الزمنية الكسرية في أنبوب. باستخدام تعريف (Caputo) للتفاضل الكسري، وبافتراض شرط إبتدائي، تم عرض حل المعادلة الواضح بالكامل وتم تمثيل الحل التحليلي بيانيا. ان طريقة التحويل الحديثة لاضطراب هوموتوبي هي مؤلفة من طريقة تحويلات لابلاس وطريقىة اضطراب هوموتوبي. ان النتائج التي تم الحصول عليها باستخدام التقنية المقترحة تدل على ان النهج هو سهل التطبيق وجذاب جدا حسابيا.

Journal of the Association of Arab Universities for Basic and Applied Sciences (2015) **17**, 14–19

University of Bahrain

Journal of the Association of Arab Universities for Basic and Applied Sciences

www.elsevier.com/locate/jaaubas
www.sciencedirect.com

ORIGINAL ARTICLE

A fractional model of Navier–Stokes equation arising in unsteady flow of a viscous fluid

Devendra Kumar [a,*], Jagdev Singh [b], Sunil Kumar [c]

[a] *Department of Mathematics, Jagan Nath Gupta Institute of Engineering and Technology, Jaipur 302022, Rajasthan, India*
[b] *Department of Mathematics, Jagan Nath University, Village-Rampura, Tehsil-Chaksu, Jaipur 303901, Rajasthan, India*
[c] *Department of Mathematics, National Institute of Technology, Jamshedpur 831014, Jharkhand, India*

Received 7 October 2013; revised 22 December 2013; accepted 14 January 2014
Available online 11 February 2014

KEYWORDS

Analytical solution;
Laplace transform method;
Homotopy perturbation method;
He's polynomials;
Fractional Navier–Stokes equations

Abstract In this paper, we present a reliable algorithm based on the new homotopy perturbation transform method (HPTM) to solve a time-fractional Navier–Stokes equation in a tube. The fractional derivative is considered in the Caputo sense. By using an initial value, the explicit solution of the equation has been presented in a closed form and then its numerical solution has been represented graphically. The new homotopy perturbation transform method is a combined form of the Laplace transform method and the homotopy perturbation method. The results obtained by the proposed technique indicate that the approach is easy to implement and computationally very attractive.

1. Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. Fractional differential equations have gained importance and popularity, mainly due to their demonstrated applications in science and engineering. For example, these equations are increasingly used to model problems in fluid flow, rheology, diffusion, relaxation, oscillation, anomalous diffusion, reaction–diffusion, turbulence, diffusive transport akin to diffusion, electric networks, polymer physics, chemical physics, electrochemistry of corrosion, relaxation processes in complex systems, propagation of seismic waves, dynamical processes in self-similar and porous structures and many other physical processes. The most important advantage of using fractional differential equations in these and other applications is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic and it is one reason why fractional calculus has become more and more popular (Caputo, 1969; Debnath, 2003; He, 1998, 1999a; Hilfer, 2000; Kilbas et al. 2006; Mainardi et al., 2001; Miller and Ross, 1993; Oldham and Spanier, 1974; Podlubny, 1999; Young, 1995).

* Corresponding author. Tel.: +91 9460905223.
E-mail addresses: devendra.maths@gmail.com (D. Kumar), jagdevsinghrathore@gmail.com (J. Singh), skumar.math@nitjsr.ac.in, skiitbhu28@gmail.com (S. Kumar).

Peer review under responsibility of University of Bahrain.

1815-3852 © 2014 Production and hosting by Elsevier B.V. on behalf of University of Bahrain.
http://dx.doi.org/10.1016/j.jaubas.2014.01.001

Recently, El-Shahed and Salem (2005) have generalized the classical Navier–Stokes equations by replacing the first time derivative by a fractional derivative of order α, $0 < \alpha \leqslant 1$. The Laplace, Fourier sine and finite Hankel transforms have been employed to obtain the exact solution for the time-fractional Navier–Stokes equations.

The Navier–Stokes equation is the primary equation of computational fluid dynamics, relating pressure and external forces acting on a fluid to the response of the fluid flow. The Navier–Stokes and continuity equations are given by:

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}.\nabla)\,\underline{u} = -\frac{1}{\rho}\nabla p + v\nabla^2 \underline{u}, \tag{1}$$

$$\nabla.\underline{u} = 0, \tag{2}$$

where ρ is the density, p is the pressure, ν is the kinematics viscosity, $\underline{u}$ is the velocity and t is the time. This model can be generalized by replacing the first-time derivative by a fractional derivative of order α, $0 < \alpha \leqslant 1$. The time-fractional model for Navier–Stokes equation then has the form of the operator equation

$$D^{\alpha}_{*t}\underline{u} + (\underline{u}.\nabla)\,\underline{u} = -\frac{1}{\rho}\nabla p + v\nabla^2 \underline{u}, \tag{3}$$

where D^{α}_{*t} denotes the Caputo fractional derivative of order α. The time-fractional Navier–Stokes equations have been studied by Ganji et al. (2010); Momani and Odibat (2006) by using the Adomian decomposition method (ADM) and the homotopy perturbation method (HPM) respectively. The homotopy perturbation method (HPM) was first introduced by He (1999b). The homotopy perturbation method has also been used by many researchers to handle linear and nonlinear problems arising in science and engineering (Ganji, 2006; He, 2003, 2006; Kumar and Singh, 2010; Kumar, 2013; Kumar et al., 2012a; Vanani et al., 2013). The homotopy analysis method was applied to study boundary layer flow in the region of the stagnation point towards a stretching sheet (Nadeem et al., 2010a) and stagnation flow of a Jeffrey fluid over a shrinking sheet (Nadeem et al., 2010b). In recent years, many authors have paid attention to study the solutions of linear and nonlinear partial differential equations by using various methods combined with the Laplace transform. Among these are the Laplace decomposition method (Gondal et al., 2013a; Khuri, 2001; Khan and Hussain, 2011; Khan and Gondal, 2012a,b; Khan et al., 2012a, 2012d, 2013; Khan, 2013), homotopy perturbation transform method (Kumar et al., 2012b, 2012c; Singh et al. 2012a, 2012b, 2013; Khan et al., 2011, 2012b, 2012e; Gondal and Khan, 2010) and homotopy analysis transform method (Arife et al., 2012; Gondal et al., 2013b; Khan et al. 2012c; Kumar et al. 2013a, 2013b; Khader et al., 2013; Salah et al., 2013).

In the present article, we consider the unsteady flow of a viscous fluid in a tube in which, besides time as one of the dependent variables, the velocity field is a function of only one space coordinate. Next, we apply the new homotopy perturbation transform method (HPTM) to solve the time-fractional Navier–Stokes equation. The homotopy perturbation transform method (HPTM) is a combination of the Laplace transform method and the homotopy perturbation method (HPM). The objective of this paper is to extend the application of the HPTM to obtain a solution of the time-fractional Navier–Stokes equation. The advantage of this technique is its capability of combining two powerful methods for obtaining exact and approximate analytical solutions for nonlinear equations. It is worth mentioning that the proposed method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach.

2. Basic definitions of fractional calculus

In this section, we mention the following basic definitions of fractional calculus.

Definition 1. The Riemann–Liouville fractional integral operator of order $\alpha > 0$, of a function $f(t) \in C_\mu, \mu \geqslant -1$ is defined as (Podlubny, 1999):

$$J^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \ \ (\alpha > 0), \tag{4}$$

$$J^0 f(t) = f(t). \tag{5}$$

For the Riemann–Liouville fractional integral we have:

$$J^{\alpha}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}t^{\alpha+\gamma}. \tag{6}$$

Definition 2. The fractional derivative of $f(t)$ in the Caputo sense is defined as (Caputo, 1969):

$$D^{\alpha}_{*t}f(t) = J^{n-\alpha}D^n f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau, \tag{7}$$

for $n-1 < \alpha \leqslant n, \ n \in N, \ \ t > 0$.

Definition 3. The Laplace transform of the Caputo derivative is given by Caputo (Caputo, 1969); see also Kilbas et al. (2006) in the form

$$L[D^{\alpha}_{*t}f(t)] = s^{\alpha}L[f(t)] - \sum_{r=0}^{n-1} s^{\alpha-r-1} f^{(r)}(0+) \ (n-1 < \alpha \leqslant n). \tag{8}$$

3. Analysis of the new proposed method

Consider unsteady, one-dimensional motion of a viscous fluid in a tube. The equations of motions which govern the flow field in the tube are the Navier–Stokes equations in cylindrical coordinates and they are given by.

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\rho \partial z} + v\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \tag{9}$$

subject to the initial condition

$$u(r,0) = f(r). \tag{10}$$

If the fractional derivative model is used to present the time derivative term, the Eq. (9) assumes the form

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = P + v\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \ 0 < \alpha \leqslant 1, \tag{11}$$

where $P = -\frac{\partial p}{\rho \partial z}$. To apply the HPTM, we write (11) in an operator form

$$D^{\alpha}_{*t} u = P + v\left(\frac{\partial^2 \mathrm{u}}{\partial \mathrm{r}^2} + \frac{1}{\mathrm{r}}\frac{\partial \mathrm{u}}{\partial \mathrm{r}}\right) \; 0 < \alpha \leqslant 1. \tag{12}$$

Applying the Laplace transform (denoted in this paper by L) on both sides of Eq. (12), we get

$$L \; [D^{\alpha}_{*t} \; u] = L[P] + L\left[v\left(\frac{\partial^2 \mathrm{u}}{\partial \mathrm{r}^2} + \frac{1}{\mathrm{r}}\frac{\partial \mathrm{u}}{\partial \mathrm{r}}\right)\right]. \tag{13}$$

Using the property of the Laplace transform, we have

$$L[u(r,t)] = \frac{P}{s^{\alpha+1}} + \frac{1}{s^{\alpha}} L\left[v\left(\frac{\partial^2 \mathrm{u}}{\partial \mathrm{r}^2} + \frac{1}{\mathrm{r}}\frac{\partial \mathrm{u}}{\partial \mathrm{r}}\right)\right]. \tag{14}$$

Operating with the Laplace inverse on both sides of Eq. (14) gives

$$u(r,t) = G(r,t) + L^{-1}\left[\frac{1}{s^{\alpha}} L\left[v\left(\frac{\partial^2 \mathrm{u}}{\partial \mathrm{r}^2} + \frac{1}{\mathrm{r}}\frac{\partial \mathrm{u}}{\partial \mathrm{r}}\right)\right]\right], \tag{15}$$

where $G(r,t)$ represents the term arising from the source term and the prescribed initial conditions. Now we apply the homotopy perturbation method

$$u(r,t) = \sum_{n=0}^{\infty} p^n \; u_n(r,t). \tag{16}$$

Substituting (16) in (15), we get

$$\sum_{n=0}^{\infty} p^n \; u_n(r,t) = G(r,t) + p(L^{-1}\left[\frac{1}{s^{\alpha}} L\left[v(\frac{\partial^2}{\partial \mathrm{r}^2}\left(\sum_{n=0}^{\infty} p^n \; u_n(r,t)\right) + \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}(\sum_{n=0}^{\infty} p^n \; u_n(r,t)))\right]\right]\Bigg), \tag{17}$$

which is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomials. Comparing the coefficients of like powers of p, the following approximations are obtained

$$p^0 : \; u_0(r,t) = G(r,t),$$

$$p^1 : \; u_1(r,t) = L^{-1}\left[\frac{1}{s^{\alpha}} L \; \left[v\left(\frac{\partial^2}{\partial \mathrm{r}^2}(u_0) + \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}(u_0)\right)\right]\right],$$

$$p^2 : \; u_2(r,t) = L^{-1}\left[\frac{1}{s^{\alpha}} L \; \left[v\left(\frac{\partial^2}{\partial \mathrm{r}^2}(u_1) + \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}(u_1)\right)\right]\right], \tag{18}$$

$$p^3 : \; u_3(r,t) = L^{-1}\left[\frac{1}{s^{\alpha}} L \; \left[v\left(\frac{\partial^2}{\partial \mathrm{r}^2}(u_2) + \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}(u_2)\right)\right]\right],$$

$$\vdots$$

4. Numerical examples

In this section, we discuss the implementation of our proposed method and investigate its accuracy by applying the homotopy perturbation method with coupling of the Laplace transform. The simplicity and accuracy of the proposed algorithm are illustrated through the following numerical examples.

Example 1. Consider the following time-fractional Navier–Stokes equation

$$\frac{\partial^{\alpha} \mathrm{u}}{\partial t^{\alpha}} = P + \frac{\partial^2 \mathrm{u}}{\partial \mathrm{r}^2} + \frac{1}{\mathrm{r}}\frac{\partial \mathrm{u}}{\partial \mathrm{r}} \; 0 < \alpha \leqslant 1, \tag{19}$$

subject to the initial condition

$$u(r,0) = 1 - r^2. \tag{20}$$

Applying the Laplace transform on both sides of Eq. (19), subject to the initial condition (20), we have

$$L[u(r,t)] = \frac{1}{s}(1 - r^2) + \frac{P}{s^{\alpha+1}} + \frac{1}{s^{\alpha}} L\left[\frac{\partial^2 \mathrm{u}}{\partial \mathrm{r}^2} + \frac{1}{\mathrm{r}}\frac{\partial \mathrm{u}}{\partial \mathrm{r}}\right]. \tag{21}$$

The inverse Laplace transform implies that

$$u(r,t) = (1 - r^2) + P\frac{t^{\alpha}}{\Gamma(\alpha+1)} + L^{-1}\left[\frac{1}{s^{\alpha}} L\left[\frac{\partial^2 \mathrm{u}}{\partial \mathrm{r}^2} + \frac{1}{\mathrm{r}}\frac{\partial \mathrm{u}}{\partial \mathrm{r}}\right]\right]. \tag{22}$$

Now applying the homotopy perturbation method, we get

$$\sum_{n=0}^{\infty} p^n \; u_n(r,t) = (1 - r^2) + P\frac{t^{\alpha}}{\Gamma(\alpha+1)} + p\left(L^{-1}\left[\frac{1}{s^{\alpha}} L\left[\frac{\partial^2}{\partial r^2}\left(\sum_{n=0}^{\infty} p^n \; u_n(r,t)\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\sum_{n=0}^{\infty} p^n \; u_n(r,t)\right)\right]\right]\right). \tag{23}$$

Comparing the coefficients of like powers of p, we have

$$p^0 : u_0(r,t) = (1 - r^2) + P\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$p^1 : u_1(r,t) = L^{-1}\left[\frac{1}{s^{\alpha}} L\left[\frac{\partial^2}{\partial \mathrm{r}^2}(u_0) + \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}(u_0)\right]\right] = -\frac{4t^{\alpha}}{\Gamma(\alpha+1)}, \tag{24}$$

$$p^2 : u_2(r,t) = L^{-1}\left[\frac{1}{s^{\alpha}} L\left[\frac{\partial^2}{\partial \mathrm{r}^2}(u_1) + \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}(u_1)\right]\right] = 0,$$

$$p^3 : u_3(r,t) = L^{-1}\left[\frac{1}{s^{\alpha}} L\left[\frac{\partial^2}{\partial \mathrm{r}^2}(u_2) + \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}(u_2)\right]\right] = 0,$$

$$\vdots$$

Therefore, the solution is

$$u(r,t) = (1 - r^2) + (P - 4)\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{25}$$

which represents the exact solution for Eq. (19) and setting $\alpha = 1$ in (25), we reproduce the solution of the classical Navier–Stokes equation as follows

$$u(r,t) = (1 - r^2) + (P - 4)t. \tag{26}$$

It is to be observed that only the third order term of the HPTM is used to evaluate the exact solution for the time-fractional Navier–Stokes equation (19). The numerical results for exact solution (25), when $\alpha = 0.5$ and $\alpha = 1$ are shown by Fig. 1(a) and (b) respectively and those for different values of r and α at $t = 1$ are depicted in Fig. 2. From Fig. 1(a) and (b), it is easy to conclude that the solution continuously depends on the time-fractional derivative.

Example 2. Consider the following time-fractional Navier–Stokes equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \quad 0 < \alpha \leqslant 1, \tag{27}$$

subject to the initial condition

$$u(r, 0) = r. \tag{28}$$

In a similar way as above, we have

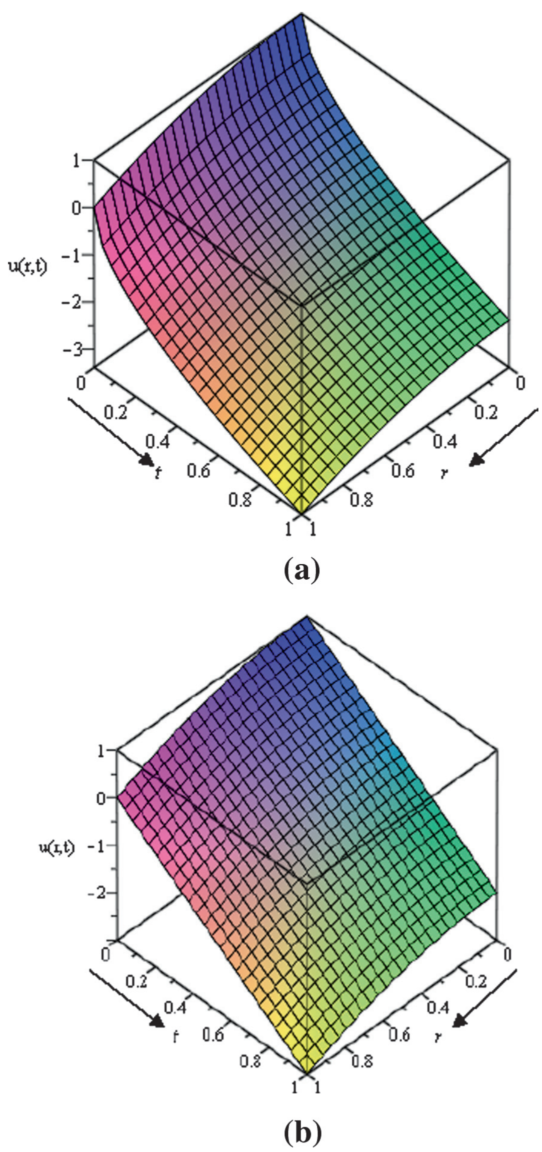

Figure 1 The surface shows the solution $u(r, t)$ for Equation (19) when $P = 1$: (a) $\alpha = 0.5$; (b) $\alpha = 1$.

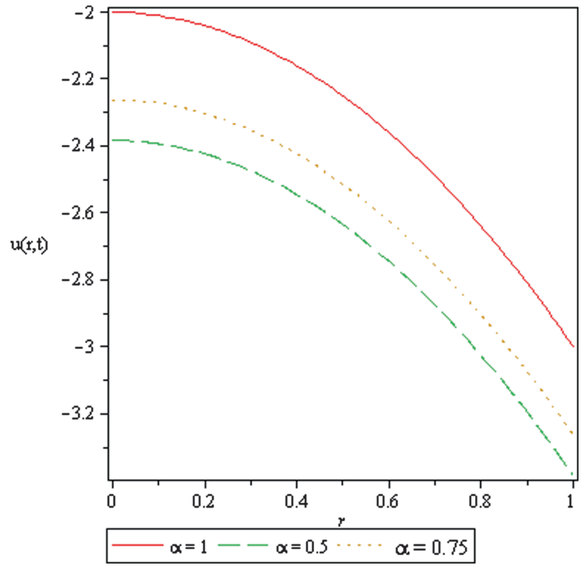

Figure 2 Plots of $u(r, t)$ vs. r at $P = 1$ and $t = 1$ for different values of α.

$$\sum_{n=0}^{\infty} p^n\, u_n(r,t) = r + p\left(L^{-1}\left[\frac{1}{s^\alpha} L\left[\frac{\partial^2}{\partial r^2}\left(\sum_{n=0}^{\infty} p^n\, u_n(r,t)\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\sum_{n=0}^{\infty} p^n\, u_n(r,t)\right)\right]\right]\right). \tag{29}$$

Comparing the coefficients of like powers of p, we have

$$\begin{aligned}
p^0 &: u_0(r,t) = r,\\
p^1 &: u_1(r,t) = \frac{1}{r}\frac{t^\alpha}{\Gamma(\alpha+1)},\\
p^2 &: u_2(r,t) = \frac{1}{r^3}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
p^3 &: u_3(r,t) = \frac{9}{r^5}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\\
&\vdots\\
p^n &: u_n(r,t) = \frac{1^2 \times 3^2 \times \cdots \times (2n-3)^2}{r^{2n-1}}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}.
\end{aligned} \tag{30}$$

Therefore, the solution is

$$u(r,t) = r + \sum_{n=1}^{\infty} \frac{1^2 \times 3^2 \times \cdots \times (2n-3)^2}{r^{2n-1}}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{31}$$

Setting $\alpha = 1$ in (31), we reproduce the solution of the problem as follows

$$u(r,t) = r + \sum_{n=1}^{\infty} \frac{1^2 \times 3^2 \times \cdots \times (2n-3)^2}{r^{2n-1}}\frac{t^n}{n!}, \tag{32}$$

which is the same solution as obtained by Baizar et al. (2002) and Momani, Odibat (2006).

5. Concluding remarks

In this paper, the new homotopy perturbation transform method (HPTM) has been successfully applied to solve the time-fractional Navier–Stokes equations in a tube with initial conditions. The technique provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation or restrictive assumptions. The results show that the solution continuously depends on the time-fractional derivative. The main advantage of this technique is to overcome the deficiency that is mainly caused by unsatisfied conditions. Thus, it can be concluded that the HPTM is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of fractional partial differential equations.

Acknowledgements

The authors are extending their heartfelt thanks to the reviewers for their valuable suggestions for the improvement of the article.

References

Arife, A.S., Vanani, S.K., Soleymani, F., 2012. The Laplace homotopy analysis method for solving a general fractional diffusion equation arising in nano-hydrodynamics. J. Comput. Theoretical Nanosci. 10, 1–4.

Baizar, J., Babalian, E., Kember, G., Nouri, A., Islam, R., 2002. An alternate algorithm for computing Adomian polynomials in special cases. Appl. Math. Comput. 138, 523–529.

Caputo, M., 1969. Elasticita e Dissipazione. Zani-Chelli, Bologna.

Debnath, L., 2003. Fractional integrals and fractional differential equations in fluid mechanics. Frac. Calc. Appl. Anal. 6, 119–155.

El-Shahed, M., Salem, A., 2005. On the generalized Navier-Stokes equations. Appl. Math. Comput. 156 (1), 287–293.

Ganji, D.D., 2006. The applications of He's homotopy perturbation method to nonlinear equation arising in heat transfer. Phys. Lett. A 335, 337–341.

Ganji, Z.Z., Ganji, D.D., Ganji, A., Rostamian, M., 2010. Analytical solution of time-fractional Navier–Stokes equation in polar coordinate by homotopy perturbation method. Numerical Methods Partial Differential Equations 26 (4), 117–124.

Gondal, M.A., Khan, M., 2010. Homotopy perturbation method for nonlinear exponential boundary layer equation using Laplace transformation, He's polynomials and Pade technology. Int. J. Non. Sci. Numer. Simul. 11, 1145–1153.

Gondal, M.A., Khan, M., Omrani, K., 2013a. A new analytical approach to two dimensional magneto-hydrodynamics flow over a nonlinear porous stretching sheet by Laplace Padé decomposition method. Results Math. 63, 289–301.

Gondal, M.A., Salah, A., Khan, M., Batool, S.I., 2013b. A novel analytical solution of a fractional diffusion problem by homotopy analysis transform method. Neural Comput. Appl. 23, 1643–1647.

He, J.H., 1998. Approximate analytic solution for seepage flow with fractional derivatives in porous media. Comput. Methods Appl. Mech. Eng. 167, 57–68.

He, J.H., 1999a. Some applications of nonlinear fractional differential equations and their approximations. Bull. Sci. Technol. 15 (2), 86–90.

He, J.H., 1999b. Homotopy perturbation technique. Comput. Methods Appl. Mech. Eng. 178, 257–262.

He, J.H., 2003. Homotopy perturbation method: a new nonlinear analytical technique. Appl. Math. Comput. 135, 73–79.

He, J.H., 2006. New interpretation of homotopy perturbation method. Int. J. Mod. Phys. B 20, 2561–2568.

Hilfer, R. (Ed.), 2000. Applications of Fractional Calculus in Physics. World Scientific Publishing Company, Singapore-New Jersey-Hong Kong, pp. 7–130.

Khan, M., Gondal, M.A., Kumar, S., 2011. A novel homotopy perturbation transform algorithm for linear and nonlinear systems of partial differential equations. World Appl. Sci. J. 12, 2352–2357.

Khan, M., Hussain, M., 2011. Application of Laplace decomposition method on semi-infinite domain. Numerical Algorithms 56, 211–218.

Khan, M., Gondal, M.A., 2012a. A reliable treatment of Abel's second kind singular integral equations. Appl. Math. Lett. 25 (11), 1666–1670.

Khan, M., Gondal, M.A., 2012b. New computational dynamics for magnetohydrodynamics flow over a nonlinear stretching sheet. Z. Naturforsch. 67a, 262–266.

Khan, M., Gondal, M.A., Batool, S.I., 2012a. A new modified Laplace decomposition method for higher order boundary value problems. Comput. Math. Organ. Theory, doi 10.1007/s10588-012-9123-0.

Khan, M., Gondal, M.A., Batool, S.I., 2012b. A Novel Analytical Implementation of Nonlinear Volterra Integral Equations. Zeitschrift fur Naturforschung 67a, 674–678.

Khan, M., Gondal, M.A., Hussain, I., Vanani, S.K., 2012c. A new comparative study between homotopy analysis transform method and homotopy perturbation transform method on semi-infinite domain. Math. Comput. Model. 55, 1143–1150.

Khan, M., Gondal, M.A., Kumar, S., 2012d. A new analytical solution procedure for nonlinear integral equations. Mathematical and Computer Modelling 55, 1892–1897.

Khan, M., Gondal, M.A., Vanani, S.K., 2012e. On the coupling of homotopy perturbation and Laplace transformation for system of partial differential equations. Appl. Math. Sci. 6, 467–478.

Khan, M., Gondal, M.A., Vanani, S.K., 2013. An efficient computational algorithm for hydromagnetic stagnation flow towards a stretching sheet. Afrika Matematika 24, 321–329.

Khan, M., 2013. A new algorithm for higher order integro-differential equations. Afrika Matematika. http://dx.doi.org/10.1007/s13370-013-0200-4.

Khader, M.M., Kumar, S., Abbasbandy, S., 2013. New homotopy analysis transform method for solving the discontinued problems arising in nanotechnology. Chin. Phys. B 22 (11), 110201.

Khuri, S.A., 2001. A Laplace decomposition algorithm applied to a class of nonlinear differential equations. J. Appl. Math. 1, 141–155.

Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., 2006. Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam.

Kumar, S., Singh, O.P., 2010. Numerical inversion of the abel integral equation using homotopy perturbation method. Z. Naturforsch. 65a, 677–682.

Kumar, S., 2013. A numerical study for solution of time fractional nonlinear Shallow-Water equation in oceans. Z. Naturforsch. A 68a, 1–7.

Kumar, S., Khan, Y., Yildirim, A., 2012a. A mathematical modelling arising in the chemical systems and its approximate numerical solution. Asia Pacific J. Chem. Eng. 7 (6), 835–840.

Kumar, S., Kocak, H., Yildirim, A., 2012b. A fractional model of gas dynamics equation by using Laplace transform. Z. Naturforsch. 67a, 389–396.

Kumar, S., Yildirim, A., Khan, Y., Leilei, W., 2012c. A fractional model of diffusion equation by using Laplace transform. Science Irantica 19 (4), 1117–1123.

Kumar, S., Singh, J., Kumar, D., Kapoor, S., 2013a. New homotopy analysis transform algorithm to solve volterra integral equation. Ain Sham, Engineering Journal, doi.org/10.1016/j.asej.2013.07.004.

Kumar, S., Kumar, D., Mahabaleshwar, U.S., 2013b. A new adjustment of Laplace transform for fractional Bloch equation in NMR flow. Application and Applied Mathematics: An International Journal (Accepted for publication).

Mainardi, F., Luchko, Y., Pagnini, G., 2001. The fundamental solution of the space-time fractional diffusion equation. Fractional Calculus Appl. Anal. 4, 153–192.

Miller, K.S., Ross, B., 1993. An Introduction to the fractional Calculus and Fractional Differential Equations. Wiley, New York.

Momani, S., Odibat, Z., 2006. Analytical solution of a time-fractional Navier–Stokes equation by Adomian decomposition method. Appl. Math. Comput. 177, 488–494.

Nadeem, S., Hussain, A., Khan, M., 2010a. HAM solutions for boundary layer flow in the region of the stagnation point towards a stretching sheet. Commun. Nonlinear Sci. Numerical Simulation 15 (3), 475–481.

Nadeem, S., Hussain, A., Khan, M., 2010b. Stagnation Flow of a Jeffrey Fluid over a Shrinking Sheet. Z. Naturforsch. 65a, 540–548.

Oldham, K.B., Spanier, J., 1974. The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order. Academic Press, New York.

Podlubny, I., 1999. Fractional Differential Equations. Academic Press, New York.

Salah, A., Khan, M., Gondal, M.A., 2013. A novel solution procedure for fuzzy fractional heat equations by homotopy analysis transform method. Neural Comput. Appl. 23 (2), 269–271.

Singh, J., Kumar, D., Sushila, 2012a. Homotopy perturbation algorithm using Laplace transform for gas dynamics equation. J. Appl. Math. Stat. Inform. 8 (1), 55–61.

Singh, J., Kumar, D., Rathore, S., 2012b. Application of homotopy perturbation transform method for solving linear and nonlinear Klein-Gordon equations. J. Inf. Comput. Sci. 7 (2), 131–139.

Singh, J., Kumar, D., Kumar, S., 2013. A new reliable algorithm for solving discontinuity problem in nanotechnology. Science Irantica 20 (3), 1059–1062.

Vanani, S.K., Yildirim, A., Soleymani, F., Khan, M., Tutkun, S., 2013. Solution of the heat equation in the cast-mould heterogeneous domain using a weighted algorithm based on the homotopy perturbation method. Int. J. Numerical Methods Heat Fluid Flow 23 (3), 451–459.

Young, G.O., 1995. Definition of physical consistent damping laws with fractional derivatives. Z. Angew. Math. Mech. 75, 623–635.