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Decay Rates of Solutions to A Nonlinear Viscoelastic Evolution Equation of Fourth Order

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ABSTRACT

In this paper we consider a nonlinear viscoelastic equation of fourth order. Under suitable conditions on the initial data and the relaxation function, we prove uniform decay results.

KEYWORDS AND PHRASES: Decay, fourth order, relaxation function, viscoelastic.

AMS Classification : 35B05, 35L05, 35L15, 35L70

1 INTRODUCTION

In this paper, we consider the asymptotic behavior of solutions to the following initial boundary

$$\begin{cases} u_{tt} - a_1 u_{xx} + b\int_0^t g(t-s)u_{xx}(s)ds - a_2 u_{xxt} \\ -a_3 u_{xxtt} = \varphi(u_x)_x, \quad x \in (0,1), \qquad t > 0 \\ u(0,t) = u(1,t) = 0, \qquad t \geq 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in [0,1], \end{cases} \tag{1}$$

value problem:
where b, ai (i = 1, 2, 3) are positive constants, φ(s) is a given nonlinear function, $u_0(x)$, $u_1(x)$ are initial value functions, g(s) is the relaxation function satisfying some conditions to be specified later. In the absence of the viscoelastic term (g = 0), this type of problems arises in the study of the strain solitary waves in nonlinear elastic rods where there exists a longitudinal wave equation of the form

$$u_{tt} - [b_0 + b_1 n(u_x)^{n-1}]u_{xx} - b_2 u_{xxtt} = 0,$$

where b_0, $b_2 > 0$ are constants, b_1 is an arbitrary real number, n is a natural number [Guitong, 1986; Zhang, Zhuang, 1988; Chen, Wang, 1995; Chen et al 2008; Komornik 1994]. Chen et al [Chen, Wang, 1995] proved that the equation with the same initial and boundary conditions given in problem (1), has a unique classical solution and gave sufficient conditions for the blow-up by the concavity method, we consider the following problem

$$\begin{cases} u_{tt} - a_1 u_{xx} - a_2 u_{xxt} - a_3 u_{xxtt} = \varphi(u_x)_x, & x \in (0,1), \quad t > 0 \\ u(0,t) = u(1,t) = 0, & t \geq 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in [0,1] \end{cases}$$

with an extra damping term of the form $a_2 u_{xxt}$ and proved the existence of a unique local solution u(x, t) C([0, T0]; C^2[0, 1]) by the contraction mapping principle. Then he used the extension theorem introduced in [Zhang, Zhuang, 1988] and proved the existence of a unique global classical solution. Later on, Chen et al, studied the following nonlinear fourth-order problem
and established a decay result. Furthermore, for initial data

$$u_0, u_1 \in H_0^1((0,1))$$

and φ C^2(IR) satisfying, for two constants A > 0, B > 0, the following condition

$$|\varphi(s)| \leq A \int_0^s \varphi(y)dy + B,$$

they made use of the integral inequality, given in [Komornik, 1994], to prove a blow up result. In the presence of the viscoelastic term, the wave equation was studied by many authors. Cavalcanti et al. [Cavalcanti et al 2002] studied

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \ t \geq 0$$

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a(x)u_t + |u|^\gamma u = 0, \quad \text{in } \Omega \times (0,\infty)$$

for □> 0, g is a positive function satisfying and a : Ω ——> IR^+ (n ≥ 1) is a function, which may vanish on a part of Ω. Under the condition that a(x) ≥ a0 > 0 on # □ Ω, with # satisfying some geometry restrictions, the authors obtained an exponential rate of decay. Berrimi and Messaoudi [Berrimi, Messaoudi, 2004] improved Cavalcanti's result by introducing a different functional, which allowed them to weaken the conditions on both *a* and *g*. In particular, the function a can vanish on the whole domain # and, consequently, no geometry condition is required. The method of proof used in [Berrimi, Messaoudi, 2004] by Berrimi and Messaoudi is based on the use of the perturbed energy technique, the choice of the Lyaponov functional made their proof easier than the one in [Cavalcanti et al 2002]. Recently, Kafini and Messaoudi [Kafini, Messaoudi, 2008] considered the following Cauchy problem.

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + u_t = |u|^{p-1}u, \quad x \in \mathbb{R}^n \quad ,t>0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases}$$

with negative initial energy and under appropriate conditions on the relaxation function, they proved a finite-time blow-up result. Actually they extended the result of [Yong, 2005], which was established for a wave equation.

In [Kafini, Messaoudi, 2007, Messaoudi, Al-jabr, 2007], Kafini and Messaoudi pushed their latter result to a coupled system of the form

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds = f_1(u,v), \quad in\ \mathbb{R}^n \times (0,\infty) \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(x,s)ds = f_2(u,v), \quad in\ \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \mathbb{R}^n. \end{cases}$$

In [Messaoudi, Al-jabr, 2007], Messaoudi et al. considered the semi linear viscoelastic equation

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + u|u|^\nu = 0, \quad \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0, \ x \in \partial\Omega \ , \qquad t \geq 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega \end{cases}$$

and showed, using a lemma by [Martinez, 1999], that the solution energy decays at a similar rate of decay of the relaxation function which is not necessarily of exponential or polynomial decay rate.

In this work, we are concerned with problem (1). Our aim is to extend the result of [Chen et 2008], established for the wave equation, to our problem. The proof is based on the multiplier method [Cavalcanti et 2001] and makes use of a lemma by [Martinez, 1999]. To achieve this goal some conditions have to be imposed on the relaxation function g.

The paper is organized as follows. In section 2, we present some notations and material needed for our work. The statement and the proof of our main result will be given in section 3.

2 PRELIMINARIES

In this section we present some material needed in the proof of our result. For this reason, let's assume that

(G1) g : $\mathbb{R}_+$ ——> $\mathbb{R}_+$ is a differentiable function such that

$$a_1 - b\int_0^\infty g(s)ds = l > 0, \qquad t \geq 0.$$

(G2) There exists a > 0 such that

$$g'(t) \leq -ag^p(t), \quad 1 \leq p < 3/2, \qquad t \geq 0.$$

(G3) φ C2(IR) and satisfies, for two constants A > 0, B > 0,

$$|\varphi(s)| \leq A\int_0^s \varphi(y)dy + B.$$

(G4) There exists a function G : IR IR such that

$$G(s) \geq 0,\ G(s) = \int_0^s \varphi(\tau)d\tau \ \ and\ \ 2G(s) \leq s\varphi(s),\ \forall s \in \mathbb{R}.$$

Now, we introduce the "modified" energy functional

$$E(t) = \frac{1}{2}\left[\int_0^1 |u_t|^2 dx + \left(a_1 - b\int_0^t g(s)ds\right)\int_0^1 |u_x|^2 dx + a_3\int_0^1 |u_{xt}|^2 dx + b(g\circ u_x)\right] + \int_0^1 G(u_x)dx, \qquad (2)$$

where

$$(g\circ u)(t) = \int_0^t g(t-s)\int_0^1 |u(s)-u(t)|^2 dxds.$$

Lemma 2.1.[Martinez, 1999]] Let E : IR+ —— > IR+ be a non-increasing function. Assume also, that there exist q ≥ 0 and A > 0 such that

$$\int_S^{+\infty} E^{q+1}(t)dt \leq AE(S), \qquad 0 \leq S < +\infty,$$

then, we have

$$E(t) \leq cE(0)e^{-\omega t} \qquad \forall t \geq 0, \qquad if \quad q = 0,$$

and

$$E(t) \leq cE(0)(1+t)^{\frac{-1}{q}} \qquad \forall t \geq 0, \qquad if \quad q > 0,$$

where c and # are positive constants independent of the initial energy E(0).

Lemma 2.2. If u is a solution of (1), then the "modified" energy satisfies

$$E'(t) = -a_2 \int_0^1 |u_{xt}|^2 \, dx - \frac{b}{2} g(t) \int_0^1 |u_x|^2 \, dx + \frac{b}{2}(g' \circ u_x) \leq 0, \tag{3}$$

which means that E(t) is a non increasing.

Proof. By multiplying the equation in (1) by ut and integrating over (0, 1), using integration by parts and repeating the same computations as in [Messaoudi, 2003], we obtain the result.

Corollary 2.3. Under the assumptions (G1) and (G2), we have

$$(g^p \circ u_x)(t) \leq c(-g' \circ u_x)(t) \leq -cE'(t). \tag{4}$$

Proof. Using (G2) and (3), the result follows.

Lemma 2.4. Let $1 < p < 3/2$ and u be the solution of (1), then for any $0 < \theta < 2-p$, there exists C such that

$$(g \circ u_x)^{\frac{p-1+\theta}{\theta}}(t) \leq C(g^p \circ u_x)(t).$$

Proof: From Lemma 3.3 [16], we have

$$(g \circ u_x)(t) \leq 2 \left\{ \left(\int_0^t g^{1-\theta}(s)ds \right) \|u_x\|_2^2 \right\}^{(p-1)/(p-1+\theta)} ((g^p \circ u_x)(t))^{\theta/(p-1+\theta)}.$$

We estimate $\int_0^t g^{1-\theta}(s)ds$, using (G1) and (G2),

$$\begin{aligned} \int_0^t g^{1-\theta}(s)ds &\leq \int_0^\infty g^{1-\theta}(s)ds \leq \frac{-1}{a} \int_0^\infty g^{1-\theta-p}(s)g'(s)ds \\ &= \frac{-1}{a(2-\theta-p)} \left[g^{2-\theta-p}(s) \right]_0^\infty < \infty. \end{aligned}$$

Also, we have

$$\sup_{0 \leq t < \infty} \|u_x\|_2^2 \leq \sup_{0 \leq t < \infty} E(t) \leq E(0).$$

Consequently, we easily see that

$$(g \circ u_x)(t) \leq C\left((g^p \circ u_x)(t)\right)^{\theta/(p-1+\theta)}.$$

3. DECAY OF SOLUTIONS

In this section, we state and prove our main result. For this purpose, we need the following lemma

Lemma 3.1. Under the assumptions (G1)-(G4) the solution of (1) satisfies, for $0 \leq S < T < \infty$,

$$\int_S^T E^{q+1}(t)dt \leq CE^{q+1}(S) + b\int_S^T E^q(t)(g \circ u_x)dt. \tag{5}$$

Proof. By multiplying equation (1) by $E^q(t)u(t)$ and integrating over $(S, T)\times(0, 1)$, we have

$$\int_S^T E^q(t)\left[\begin{array}{c} \int_0^1 uu_{tt}dx - a_1\int_0^1 uu_{xx}dx + b\int_0^1 u(t)\int_0^t g(t-s)u_{xx}(s)dsdx \\ -a_2\int_0^1 uu_{xxt}dx - a_3\int_0^1 uu_{xxtt}dx \end{array}\right]dt$$
$$= \int_S^T E^q(t)\int_0^1 u\varphi(u_x)_x dxdt.$$

A direct integration by parts yields

$$\int_S^T E^q(t)\left(\int_0^1 (a_1u_x^2 - u_t^2 - a_3u_{xt}^2)dx\right)dt$$
$$+b\int_S^T E^q(t)\left(\int_0^t g(t-s)\int_0^1 u(t)u_{xx}(s)dxds\right)dt$$

$$-q\int_S^T E^{q-1}E'(t)\left(\int_0^1 uu_tdx + a_2\int_0^1 u_x^2dx + a_3\int_0^1 u_{xt}u_xdx\right)dt$$
$$= -\left[E^q(t)\int_0^1 uu_tdx\right]_S^T - \frac{a_2}{2}\left[E^q(t)\int_0^1 u_x^2dx\right]_S^T - a_3\left[E^q(t)\int_0^1 u_{xt}u_xdx\right]_S^T$$
$$-\int_S^T E^q(t)\int_0^1 u_x\varphi(u_x)dxdt. \tag{6}$$

The integrand of the second term in the left side of (6) can be estimated as follows

$$\begin{aligned}
& bE^q(t)\int_0^t g(t-s)\int_0^1 u(t)u_{xx}(s)dxds \\
= & -bE^q(t)\int_0^t g(t-s)\int_0^1 u_x(t)u_x(s)dxds \\
= & -bE^q(t)\left(\int_0^t g(t-s)\int_0^1 u_x(t)[u_x(s)-u_x(t)]dxds+\int_0^t g(t-s)\int_0^1 u_x(t)u_x(t)dxds\right) \\
= & -bE^q(t)\left(\int_0^1 u_x(t)\int_0^t g(t-s)[u_x(s)-u_x(t)]dxds+\int_0^t g(t-s)\int_0^1 |u_x(t)|^2 dxds\right) \\
= & -bE^q(t)\int_0^1 u_x(t)\int_0^t g(t-s)[u_x(s)-u_x(t)]dxds-bE^q(t)\int_0^t g(t-s)\int_0^1 |u_x(t)|^2 dxds \\
\geq & -bE^q(t)\left(\delta\int_0^1 |u_x(t)|^2 dx+\frac{1}{4\delta}\int_0^1\left(\int_0^t g^{2-p}(s)ds\right)\left(\int_0^t g^p(t-s)\,|u_x(t)-u_x(s)|^2 ds\right)dx\right) \\
& -bE^q(t)\int_0^t g(t-s)\int_0^1 |u_x(t)|^2 dxds \\
\geq & -bE^q(t)\left(\delta\int_0^1 |u_x(t)|^2 dx+\frac{1}{4\delta}\left(\int_0^t g^{2-p}(s)ds\right)(g^p\circ u_x)(t)\right) \\
& -bE^q(t)\int_0^t g(s)ds\int_0^1 |u_x(t)|^2 dx.
\end{aligned}$$

Thus (6) becomes

$$\begin{aligned}
& \int_S^T E^q(t)\int_0^1\left[\left(a_1-b\delta-b\int_0^t g(s)ds\right)|u_x|^2-|u_t|^2-a_3\,|u_{xt}|^2\right]dxdt \\
& -\frac{b}{4\delta}\left(\int_0^t g^{2-p}(s)ds\right)\int_S^T E^q(t)(g^p\circ u_x)dt \\
& -q\int_S^T E^{q-1}E'(t)\left(\int_0^1 uu_t dx+a_2\int_0^1 u_x^2dx+a_3\int_0^1 u_{xt}u_x dx\right)dt \\
\leq & -\left[\int_0^1 E^q(t)uu_t dx\right]_S^T-\frac{a_2}{2}\left[\int_0^1 E^q(t)u_x^2dx\right]_S^T-a_3\left[\int_0^1 E^q(t)u_{xt}u_x dx\right]_S^T \\
& -\int_S^T E^q(t)\int_0^1 u_x\varphi(u_x)dxdt. \qquad (7)
\end{aligned}$$

Adding $\int_S^T E^q(t) \int_0^1 2G(u_x)dxdt$ to both sides of (7), we get

$$\begin{aligned} & 2\int_S^T E^{q+1}(t)dt \\ \leq\ & 2\int_S^T E^q(t)\int_0^1 \left[|u_t|^2 + \frac{b\delta}{2}|u_x|^2 + a_3|u_{xt}|^2\right]dxdt \\ & +\frac{1}{4\delta}\left(\int_0^t g^{2-p}(s)ds\right)\int_S^T E^q(t)(g^p \circ u_x)dt \\ & -q\int_S^T E^{q-1}E'(t)\left(\int_0^1 uu_t dx + a_2\int_0^1 u_x^2 dx + a_3\int_0^1 u_{xt}u_x dx\right)dt \\ & -\left[E^q(t)\int_0^1 uu_t dx\right]_S^T - \frac{a_2}{2}\left[E^q(t)\int_0^1 u_x^2 dx\right]_S^T - a_3\left[E^q(t)\int_0^1 u_{xt}u_x dx\right]_S^T \\ & +b\int_S^T E^q(t)(g\circ u_x)dt. \end{aligned} \quad (10)$$

$$+\int_S^T E^q(t)\int_0^1 \left(2G(u_x) - u_x\varphi(u_x)\right)dxdt. \quad (9)$$

Exploiting (2), estimate (8) takes the form
By virtue of assumption (G4), we get from (9)

Terms in the right side of (10) can be estimated, using the nonincreasing property of E(t), Poincaré's inequality, Cauchy's inequality, assumption (G2), Corollary 2.3 and the definition of E(t), as follows the first term

$$\begin{aligned} & \int_S^T E^q(t)\int_0^1 \left[\left(a_1 - b\delta - b\int_0^t g(s)ds\right)|u_x|^2 - |u_t|^2 - a_3|u_{xt}|^2\right]dxdt \\ & -\frac{1}{4\delta}\left(\int_0^t g^{2-p}(s)ds\right)E^q(t)(g^p\circ u_x)(t) + \int_S^T E^q(t)\int_0^1 2G(u_x)dxdt \\ & -q\int_S^T E^{q-1}E'(t)\left(\int_0^1 uu_t dx + a_2\int_0^1 u_x^2 dx + a_3\int_0^1 u_{xt}u_x dx\right)dt \\ \leq\ & -\left[\int_0^1 uu_t dx\right]_S^T - \frac{a_2}{2}\left[\int_0^1 u_x^2 dx\right]_S^T - a_3\left[\int_0^1 u_{xt}u_x dx\right]_S^T \\ & +\int_S^T E^q(t)\int_0^1 (2G(u_x) - u_x\varphi(u_x))dxdt. \end{aligned} \quad (8)$$

$$\begin{aligned}2\int_S^T E^q(t)\int_0^1 |u_t|^2\,dxdt &\le 2\int_S^T E^q(t)\int_0^1 |u_{xt}|^2\,dxdt \le -c\int_S^T E^q(t)E'(t)dt\\ &= c[E^{q+1}(S)-E^{q+1}(T)] \le cE^{q+1}(S),\end{aligned}$$

the second term

$$b\delta\int_S^T E^q(t)\int_0^1 |u_x|^2\,dxdt \le \delta c\int_S^T E^{q+1}(t)dt,$$

the third term

$$2a_3\int_S^T E^q(t)\int_0^1 |u_{xt}|^2\,dxdt \le -c\int_S^T E^q(t)E'(t)dt = c[E^{q+1}(S)-E^{q+1}(T)] \le cE^{q+1}(S),$$

the fourth term

$$\begin{aligned}&\frac{1}{4\delta}\left(\int_0^t g^{2-p}(s)ds\right)\int_S^T E^q(t)(g^p\circ u_x)dt\\ \le\ &\frac{c}{4\delta}\int_S^T E^q(t)(g^p\circ u_x)dt \le \frac{-c}{4\delta}\int_S^T E^q(t)(g'\circ u_x)dt\\ \le\ &\frac{-c}{4\delta}\int_S^T E^q(t)E'(t)dt \le \frac{c}{4\delta}[E^{q+1}(S)-E^{q+1}(T)]\\ \le\ &\frac{c}{4\delta}E^{q+1}(S),\end{aligned}$$

where, by assumption G(2), we used the fact that

$$\int_0^t g^{2-p}(s)ds < \int_0^\infty g^{2-p}(s)ds < \infty,$$

the fifth term

$$\begin{aligned}&-q\int_S^T E^{q-1}E'(t)\int_0^1 uu_t\,dxdt\\ \le\ &-c\int_S^T E^{q-1}E'(t)\int_0^1\left(|u|^2+|u_t|^2\right)dxdt\\ \le\ &-c\int_S^T E^{q-1}E'(t)\int_0^1\left(|u_x|^2+|u_{xt}|^2\right)dxdt\\ \le\ &-c\int_S^T E^{q-1}E'(t)E(t)dt \le -c\int_S^T E^q(t)E'(t)dt\\ \le\ &c[E^{q+1}(S)-E^{q+1}(T)] \le cE^{q+1}(S),\end{aligned}$$

the sixth term

$$\begin{aligned} & -a_2 q \int_S^T E^{q-1} E'(t) \int_0^1 u_x^2 dx \\ \leq & -c \int_S^T E^{q-1} E'(t) E(t) dt \leq -c \int_S^T E^q(t) E'(t) dt \\ \leq & c[E^{q+1}(S) - E^{q+1}(T)] \leq c E^{q+1}(S), \end{aligned}$$

the seventh term

$$\begin{aligned} & -a_3 q \int_S^T E^{q-1} E'(t) \int_0^1 u_{xt} u_x dx dt \\ \leq & -c \int_S^T E^{q-1} E'(t) \int_0^1 \left(|u_{xt}|^2 + |u_x|^2 \right) dx dt \\ \leq & -c \int_S^T E^{q-1} E'(t) E(t) dt \leq -c \int_S^T E^q(t) E'(t) dt \\ \leq & c[E^{q+1}(S) - E^{q+1}(T)] \leq c E^{q+1}(S), \end{aligned}$$

the eighth term

$$\begin{aligned} \left| E^q(t) \int_0^1 u u_t dx \right| & \leq \frac{1}{2} E^q(t) \int_0^1 \left(|u|^2 + |u_t|^2 \right) dx \\ & \leq \frac{1}{2} E^q(t) \int_0^1 \left(|u_x|^2 + |u_{xt}|^2 \right) dx \leq c E^{q+1}(t), \end{aligned}$$

the ninth term

$$\frac{a_2}{2} E^q(t) \int_0^1 |u_x|^2 dx \leq c E^{q+1}(t),$$

the tenth term

$$\left| a_3 E^q(t) \int_0^1 u_{xt} u_x dx \right| \leq \frac{a_3}{2} E^q(t) \int_0^1 \left(|u_{xt}|^2 + |u_x|^2 \right) dx \leq c E^{q+1}(t).$$

By inserting the above estimates in (10) and taking * small enough, the result follows. Now, we are in position to state and prove our main result.

Theorem 3.2.

$$\textit{Let } (u_0, u_1) \in H_0^1((0,1)) \times H_0^1((0,1)) \textit{ be given.}$$

Assume that (G1) - (G4) hold. Then, there exist strictly positive constants C and ω such that, for any $t \geq 0$,

$$
\begin{aligned}
E(t) &\leq Ce^{-\omega t}, \qquad if \quad p = 1, \\
E(t) &\leq C(1+t)^{\frac{-1}{p-1}}, \quad if \quad p > 1.
\end{aligned}
$$

Proof. (Case p = 1)
Estimates (5) and (4) yield

$$\int_S^T E^{q+1}(t)dt \leq AE^{q+1}(S), \quad \forall S \geq 0,$$

take q = 0, we get

$$\int_S^T E(t)dt \leq AE(S), \quad \forall t \geq 0,$$

then let T ——> +∞ to get

$$\int_S^{+\infty} E(t)dt \leq AE(S), \quad \forall t \geq 0, \tag{11}$$

thus Lemma 2.1 yields

$$E(t) \leq cE(0)e^{-\omega t} \leq Ce^{-\omega t}, \quad \forall t \geq 0.$$

(Case p > 1)
Lemma 2.4 and Young's inequality, for 0 < θ < 1, yield

$$
\begin{aligned}
& E^q(t)(g \circ u_x) \\
\leq \quad & cE^q(t)(g^p \circ u_x)^{\frac{\theta}{p-1+\theta}} \leq c\left[\varepsilon E^{\frac{q(p-1+\theta)}{p-1}}(t) + c(\varepsilon)(g^p \circ u_x)\right],
\end{aligned}
$$

we choose $q = \frac{p-1}{\theta}$ so that $\frac{q(p-1+\theta)}{p-1} = q + 1$.
Consequently, we obtain from (5),

$$\int_S^T E^{q+1}(t)dt \leq AE(S), \quad \forall S \geq 0,$$

then let T ——> +∞ to get

$$\int_S^{+\infty} E^{q+1}(t)dt \leq AE(S), \quad \forall S \geq 0.$$

Thus Lemma 2.1 yields

$$E(t) \leq cE(0)(1+t)^{\frac{-1}{q}} = C(1+t)^{\frac{-\theta}{p-1}}, \quad \forall t \geq 0. \tag{12}$$

Using (12), we estimate

$$\int_0^t \left(\int_0^1 |u_x|^2\, dx\right) ds \le c\int_0^\infty E(t)dt \le c\int_0^\infty (1+t)^{\frac{-\theta}{p-1}}dt \le c\left[(1+t)^{\frac{p-\theta-1}{p-1}}\right]_0^\infty,$$

and we choose $\theta = \frac{1}{2}$ so that $p - \theta - 1 < \frac{3}{2} - \frac{1}{2} - 1 = 0$.

Consequently, we obtain

$$\int_0^t \left(\int_0^1 |u_x|^2\, dx\right) ds < \infty,$$

and

$$\sup_{0\le t<\infty} t\left(\int_0^1 |u_x|^2\, dx\right) \le \sup_{0\le t<\infty} ctE(t) \le \sup_{0\le t<\infty} ct(1+t)^{\frac{-\theta}{p-1}} \le \sup_{0\le t<\infty} \frac{ct}{(1+t)^{\frac{\theta}{p-1}}} < \infty,$$

thus Lemma 3.3 [16] yields

$$(g^p \circ u_x) \ge c(g\circ u_x)^p. \tag{13}$$

Using (13), we have the following estimate

$$\begin{aligned} E^q(t)(g\circ u_x) &\le cE^q(t)(g^p\circ u_x)^{\frac{1}{p}} \\ &\le c\left[\varepsilon\left(E^q\right)^{\frac{p}{p-1}}(t) + c(\varepsilon)\left(\left(g^p\circ u_x\right)^{\frac{1}{p}}\right)^p\right] \\ &\le c\left[\varepsilon E^{\frac{pq}{p-1}}(t) + c(\varepsilon)(g^p\circ u_x)\right], \end{aligned} \tag{14}$$

then we choose $q = p - 1$ so that $\frac{pq}{p-1} = q + 1$.

Consequently, from (14), we obtain

$$E^q(t)(g\circ u_x) \le c\left[\varepsilon E^{q+1}(t) + c(\varepsilon)(-E'(t))\right],$$

integration over (S, T), gives

$$\int_S^T E^q(t)(g\circ u_x)dt \le c\left[\varepsilon\int_S^T E^{q+1}(t)dt + c(\varepsilon)E(S)\right]. \tag{15}$$

For small enough, estimate (15) yields

$$\int_S^T E^{q+1}(t)dt \leq CE(S), \quad \forall S \geq 0.$$

As T ——> +∞, we obtain

$$\int_S^{+\infty} E^{q+1}(t)dt \leq AE(S), \quad \forall S \geq 0.$$

Thus, Lemma 2.1 yields

$$E(t) \leq cE(0)(1+t)^{\frac{-1}{q}} \leq C(1+t)^{\frac{-1}{p-1}}, \qquad \forall t \geq 0.$$

This completes the proof.

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معدلات أضمحلال الحلول المعادلة تطوير المرونة اللزوجة غير الخطية من الرتبة الرابعة

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الملخص

وفي ظل شروط مناسبة حلول البيانات الأولية والدالة المتراخية برهنا نتائج أضمحلال منتظم.