



Shape Measures based on Mean Absolute Deviation with Graphical Display

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Abstract: Mean absolute deviation about median is partitioned into two parts in terms of median to obtain a measure of skewness that is zero for symmetric distributions and into four parts in terms of percentiles to obtain a measure of equality between the middle and the sides of a distribution that is zero for the normal distribution. Based on these partitions a powerful and informative graph called H-graph is produced that can provide more insight into the nature of the data and assess goodness of fit for a data set to a theoretical model. This graph enriches the visual information offered by the histogram and box-plot. By using these measures two tests for middle-sides equality and normality are proposed. The simulation results from several distributions show that the proposed tests have a very good power in comparisons with well-known powerful tests that depend on moments.

Keywords: box-plot, kurtosis, normality test, MAD, skewness.

1. INTRODUCTION

The shape of a distribution may be considered either descriptively, using terms such as "U-shaped", or numerically, using quantitative measures such as skewness and kurtosis. Considerations of the shape of a distribution arise in statistical data analysis where simple quantitative descriptive statistics and plotting techniques such as histogram can lead to the selection of a particular family of distributions for modeling purposes. The shape of a distribution is sometimes characterized by the behavior of the tails as in a long or short tail; see, Balanda and MacGillivray (1988), DeCarlo (1997) and Thode (2002) and Tukey (1977). Pearson (1905) referred to leptokurtic distributions as being more peaked and platykurtic distributions as being less peaked than normal distribution. According to van Zwet (1964) only symmetric distributions should be compared in terms of kurtosis. The interest in assessing shape of the distributions may be due to the increasing use of normal theory covariance structure methods which are known to perform poorly in asymmetric and leptokurtic distributions (Hu et al., 1990 and Micceri, 1989), nonparametric tests of location such as the Mann-Whitney test can be far more powerful than the t-test in certain leptokurtic distributions (Hodges and

Lehmann,1956) and many variables show platykurtic such as the time between eruptions of certain geysers, the color of galaxies and the size of worker weaver ant.

Mean absolute deviation about median (MAD_{med}) is divided to two parts in terms of median to obtain a measure of skewness that is zero for symmetric distributions and to four parts in terms of 12th, 50th and 88th percentiles to obtain a measure of middle-sides equality that is zero for normal distribution. Based on these partitions an informative graph called H-graph is presented that can provide more insight into the nature of the data and make an assessment for a data set to a theoretical distribution to find out if the assumption of a common distribution is justified. Based on these measures two tests for middle-sides equality and normality are proposed. The simulation study is conducted to obtain and compare the empirical Type I error and the power of the proposed tests with Anscombe-Glynn, Bonett-Seier and Jarque-Bera tests from several distributions.

In Section 2 the MAD_{med} is divided to two and four parts based on percentiles. In Section 3 the measure of skewness, peak-tail equality and H-graph are introduced. The estimation of skewness and middle-sides equality measures is presented in Section 4. The middle-sides

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equality and omnibus normality tests are studied in section 5. Section 6 is devoted to the conclusion.

2. PARTITIONS OF MAD_{MED}

Let X_1, X_2, \dots, X_n be a random sample from a continuous distribution with density function $f(x)$, quantile function $x(F) = F^{-1}(x) = Q(F)$, $0 < F < 1$, cumulative distribution function $F(x) = F$, mean $\mu = E(X)$ and median $\nu = Med(X)$. The population MAD_{med} is defined as

$$D = E|X - \nu| \quad (1)$$

The MAD_{med} can be partitions to two parts above and below the median ($\nu = x(50\%)$), see Seier and Bonett (2011), as

$$D = E|X - \nu| = E(X - \nu)_{X > \nu} + E(\nu - X)_{X < \nu} = D^+ + D^- \quad (2)$$

Also for $\nu_3 > \nu > \nu_1$, the MAD_{med} can be partitions to four parts below ν_1 , between ν_1 and ν , between ν and ν_3 and above ν_3 as

$$D = E|X - \nu| = E(\nu - X)_{X < \nu_1} + E(\nu - X)_{\nu_1 < X < \nu} + E(X - \nu)_{\nu < X < \nu_3} + E(X - \nu)_{X > \nu_3} = D^{\nu_1} + D^{(\nu_1, \nu)} + D^{(\nu, \nu_3)} + D^{\nu_3} \quad (3)$$

The integral form representations of equations (2) and (3) could be written as

$$D^+ = \int_0^1 (X - \nu)I(X > \nu)dF(x),$$

$$D^- = \int_0^1 (\nu - X)I(X < \nu)dF(x),$$

$$D^{\nu_1} = \int_0^1 (\nu - X)I(X < \nu_1)dF(x),$$

$$D^{(\nu_1, \nu)} = \int_0^1 (\nu - X)I(\nu_1 < X < \nu)dF(x),$$

$$D^{(\nu, \nu_3)} = \int_0^1 (X - \nu)I(\nu < X < \nu_3)dF(x),$$

$$D^{\nu_3} = \int_0^1 (X - \nu)I(X > \nu_3)dF(x),$$

and

$$D^{(\nu_1, \nu_3)} = \int_0^1 |X - \nu|I(\nu_1 < X < \nu_3)dF(x),$$

The main advantage of the MAD it is uniquely characterize the probability distribution where Perez and Gomez (1990) said that "the dispersion function defined as $D(u) = E|X - u|$, $u \in R$ characterizes the distribution function and gives a dispersive ordering of probability distributions..."

3. SHAPE MEASURES USING MAD_{MED}

A. Skewness measure and H-graph

The skewness measure based on partitions of MAD_{med} is

$$S = \frac{D^+ - D^-}{D} = H^+ - H^- \quad (4)$$

This measure is zero for any symmetric distribution and is bounded by -1 and 1 . This measure is equivalent to $(\mu - Q_2)/E|Y - Q_2|$ which derived by Groeneveld and Meeden (1984) who have put forward the following four properties that any reasonable coefficient of skewness $S(y)$ should satisfy: (1) for any $a > 0$ and real b , $S(y) = S(ay + b)$; (2) if y is symmetrically distributed, then $S(y) = 0$; (3) $-S(y) = S(-y)$; (4) if F and G are cumulative distribution functions of y and x , and $F <_c G$, then $S(y) \leq S(x)$ where $<_c$ is a skewness-ordering among distributions; see van Zwet (1964). The measure S satisfies the four properties as pointed out by Groeneveld and Meeden (1984). The measure S can be shown graphically on the H-graph that shows the index of the order data on x-axis and $Y_{i:n} = (X_{i:n} - \nu)/D$ on y-axis. Note that $X_{i:n}$ can be theoretically represented by $Med(X_{i:n}) = x(Med F) = x\left(\frac{i-0.3175}{n+0.365}\right)$; see, Filliben (1975). Therefore, H^- represents the standardized expected value of the heights between the line at $Y = 0$ and the curve for the values less than the median and H^+ represents the standardized expected value of the heights between the line at $Y = 0$ and the curve for the values more than median.

Figure 1 shows the H-graph for the normal, uniform, exponential and beta distributions. The graph shows symmetric H 's areas with medium tails for the normal distribution and short and fat tails for the uniform distribution (zero skewness) while much more H^+ than H^- with long and slim right tail for the exponential distribution (positive skewness) and much more H^- than H^+ with medium and fat left tail for the beta(0.5,0.2) distribution (negative skewness).

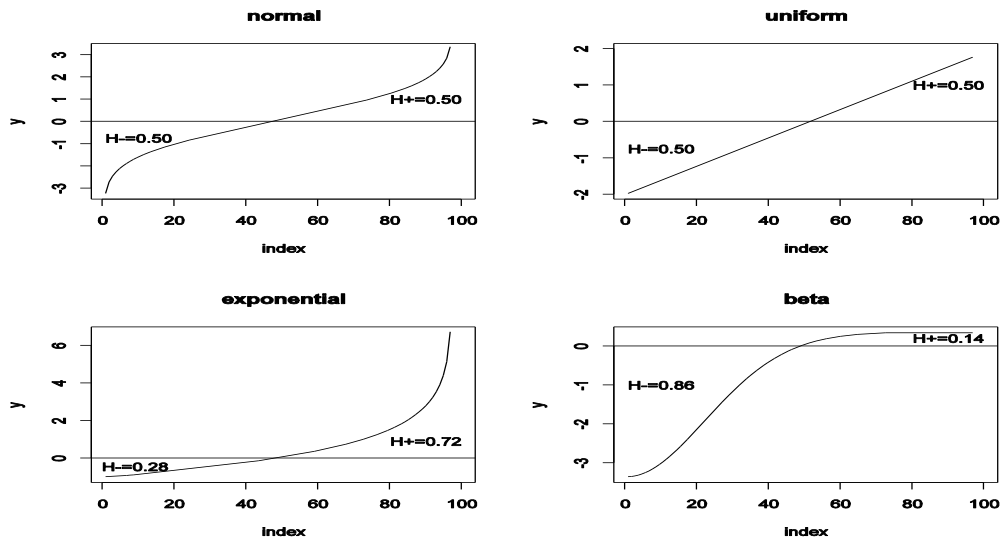


Figure 1. H^- and H^+ on H -graph for normal ($S = 0$), uniform ($S = 0$), exponential ($S = 0.44$) and beta(0.5,0.2) ($S = -0.72$) distributions.

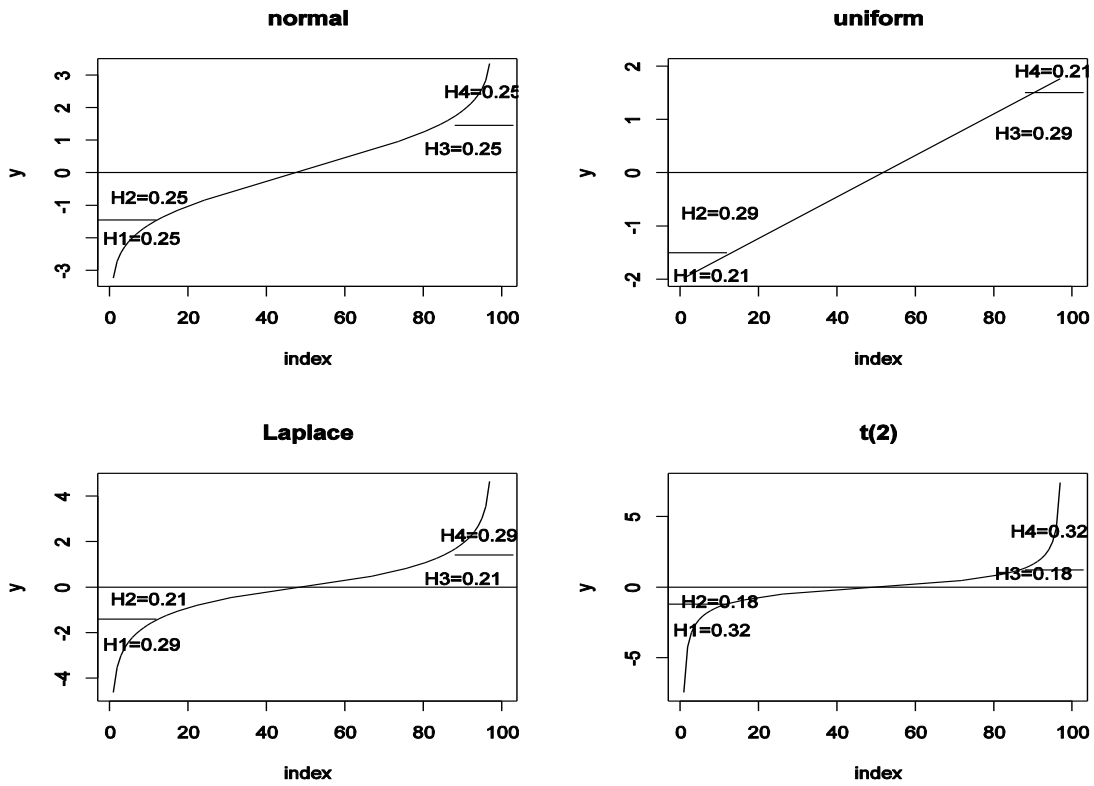


Figure 2. H_1 , H_2 , H_3 and H_4 on H -graph for normal ($K = 0$), uniform ($K = -0.155$), Laplace ($K = 0.165$) and $t(2)$ ($K = 0.28$) distributions.



B. Middle-side equality measure and H-graph

The shape proposed middle-sides equality measure based on partitions of MAD_{med} is defined as

$$\begin{aligned}
 K &= \frac{D^{v_1} - (D^{(v_1, v)} + D^{(v, v_3)}) + D^{v_3}}{D} \\
 &= H_1 - (H_2 + H_3) + H_4 \\
 &= (H_1 - H_2) + (H_4 - H_3)
 \end{aligned}
 \tag{5}$$

or equivalently

$$K = \frac{D^{v_1} - D^{(v_1, v_3)} + D^{v_3}}{D} = H_1 - H_{23} + H_4
 \tag{5}$$

This measure is bounded by -1 and 1 for all distributions and the choice of $v_1 = x(11.952\%)$ and $v_3 = x(88.048\%)$ to obtain middle-sides equality measure equal to approximately zero for the normal distribution. Table 1 below gives the results for the values of K for different percentile from standard normal distribution using quantile $\frac{i-0.3175}{n+0.365}$ and $n = 500000$.

Table 1. Values of K from standard normal distribution for different percentile

Percentile	K
p=0.08, 1-p=0.92	-0.25470
p=0.10, 1-p=0.90	-0.12018
p=0.11, 1-p=0.89	-0.05733
p=0.115, 1-p=0.885	-0.02691
p=0.119, 1-p=0.881	-0.00305
P=0.1194, 1-p=0.8806	-0.00069
p=0.1195, 1-p=0.8805	-0.00009
p=0.11951, 1-p=0.88049	-0.00004
p=0.11952, 1-8=0.88048	0.00002
p=0.1196, 1-p=0.8804	0.00049
p=0.12, 1-p=0.88	0.00285
p=0.125, 1-p=0.875	0.03199
p=0.13, 1-p=0.87	0.06052
p=0.15, 1-p=0.85	0.16888

Therefore, H_1 represents the standardized expected value of the MAD_{med} for the values less than v_1 or the heights between the line at $Y = v_1$ and the curve for the values

less than v_1 , H_2 represents the standardized expected value of the MAD_{med} for the values more than v_1 and less than v , or the heights between the lines at $Y = v_1$, median and the curve, H_3 represents the standardized expected value of the MAD_{med} for the values more than v and less than v_3 , or the heights between the lines at $Y = v_3$, median and the curve, H_4 represents the standardized expected value of the MAD_{med} for the values more than v_3 or the heights between the line at $Y = v_3$ and the curve for the values more than v_3 . Therefore, $H_1 + H_4$ can be interpreted as the probability mass that concentrated in the sides of the distribution (sides mass) in terms of MAD_{med} while $H_2 + H_3 = H_{23}$ can be interpreted as the probability mass that concentrated in the middle of the distribution (middle mass) in terms of MAD_{med} , i.e. the K measure compares the sides mass with middle mass in terms of MAD_{med} and with respect to the normal distribution, therefore, if $K = 0$, the sides mass equal to peak mass (middle-sides equality), $K > 0$ then sides mass is more than middle mass or heavier sides mass and lighter middle mass than normal (sides mass) and $K < 0$ then the sides mass is less than middle mass or lighter sides mass and heavier middle mass than normal (middle mass).

Table 2. Values of S and K for some symmetric distributions

Set A	S	K	Set B	S	K
Beta(0.25,0.25)	0	-	gl*(0,1,-0.85,-0.85)	0	0.653
Beta(0.5,0.5)	0	-	gl(0,1,-0.75,-0.75)	0	0.568
Uniform	0	-	gl(0,1,-0.5,-0.5)	0	0.360
Beta(1.5,1.5)	0	-	gl(0,1,-0.25,-0.25)	0	0.190
Normal	0	0	gl(0,1,-0.15,-0.15)	0	0.130
Logistic	0	0.056	gl(0,1,-0.10,-0.10)	0	0.105
Laplace	0	0.165	gl(0,1,-0.05,-0.05)	0	0.081

*gl stands for generalized lambda distribution with four parameters;see, Ramberg et al. (1979)

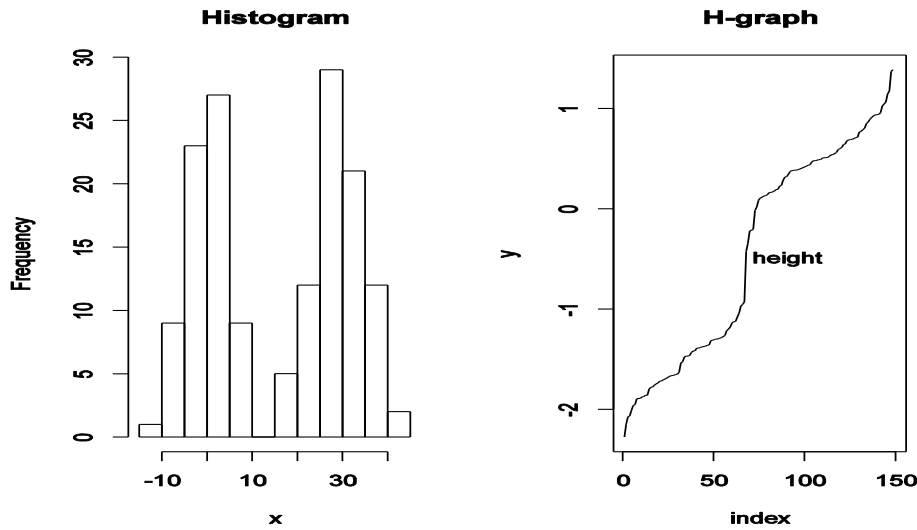


Figure 3. The histogram and H-graph for bi-modal data and H-graph shows zigzag curve with one height (bimodal distribution)

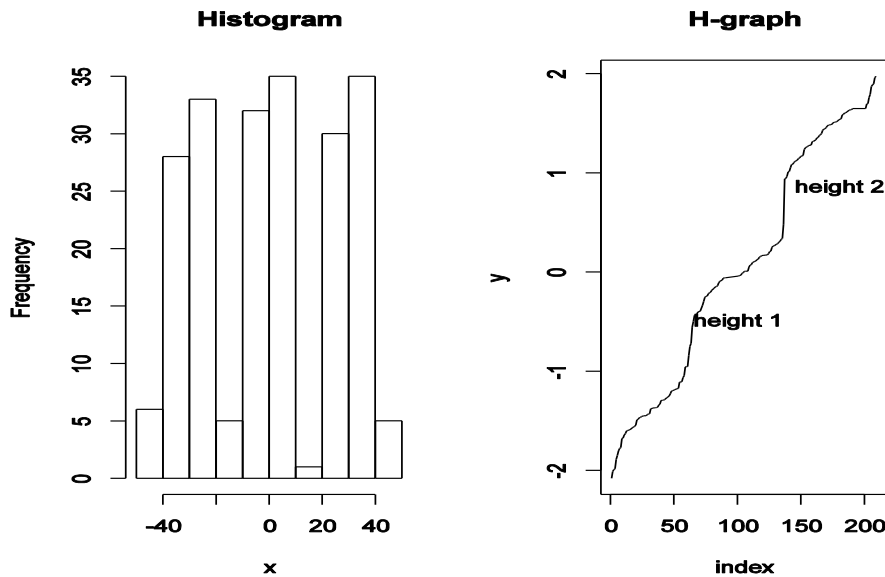


Figure 4. The histogram and H-graph for data with three modes and H-graph shows zigzag curve with two heights (tri-modal distribution)

Figure 2 shows the H -graph for the normal, uniform, Laplace and $t(2)$ distributions. The graph shows equal H 's areas (0.25) for the normal distribution and in this case $K = 0.5 - 0.5 = 0$ (middle-sides equality), for the uniform distribution the middle mass (0.58) is more than the sides mass (0.42) and in this case $K = 0.42 - 0.58 = -0.16$ (middle mass). For Laplace distribution the middle mass (0.42) is less than the sides mass (0.58) and in this case $K = 0.58 - 0.42 = 0.16$ (sides mass) while for the $t(2)$ distribution the middle mass (0.36) is much less than the sides mass (0.64) and in this case $K = 0.64 - 0.36 = 0.28$ (sides mass) with respect to normal distribution.

DeCarlo (1997) and others have pointed out that the Laplace distribution is clearly more peaked than the t_5 distribution but the classical shape measure (Pearson's kurtosis measure) $\beta_2 = 6$ for the Laplace and $\beta_2 = 9$ for the t_5 . In contrast, $K = 0.165$ for the Laplace and $K = 0.088$ for the t_5 and thus K correctly classifies these distributions according to middle mass. Note that K is a location and scale invariant and rank the distributions in Set A from smallest to largest and exists in distributions where $x(50\%)$, $x(11.952\%)$ and $x(88.048\%)$ exist while β_2 exists in distributions where fourth moment exists. Therefore it may be considered K as a measure of kurtosis where it is according to Oja



(1981), a valid measure of kurtosis must be location and scale invariant and also must obey van Zwet ordering which rank orders the distributions in Set A of Table 2 from smallest to largest.

C. H-graph and multimodality

The zigzag terms of original data. Figure 6 shows the H-graph for 100 observations from normal distribution with mean 80 and standard deviation It is not necessarily the H-graph will be plotted in standardized data but also it could be plotted in terms of original data. Figure 6 shows the H-graph for 100 observations from normal distribution with Figure 5 shows that the blue distribution is asymmetric to the right and has long and heavy right tail. Moreover the two distributions are almost Figure 5 shows that the blue distribution is asymmetric to the right and has long and heavy right tail. Moreover the two distributions are almost the same in middle mass ($H_2 + H_3$) but they are very different in sides mass ($H_1 + H_4$) and both are unimodal distributions where the curves are smooth.

Figure 3 shows 2 bends with one height that indicates bimodal distribution while Figure 4 shows 3 bends with two heights that indicates tri-modal distribution.

Moreover, when there are two data samples or to compare a data set to a theoretical model to know if the assumption of a common distribution is justified. The H-graph can provide more insight into the nature of the difference and an assessment of goodness of fit that is a graphical method rather than reducing to a numerical summary in terms of skewness, kurtosis, middle mass, sides mass and modality.

It is not necessarily the H-graph will be plotted in standardized data but also it could be plotted in terms of original data. Figure 6 shows the H-graph for 100 observations from normal distribution with mean 80 and standard deviation 10. The graph reflects a lot of information such as min, max, third, second, first quartiles and shapes.

4. ESTIMATION

We now consider estimators of population MAD_{med} using a random sample of size n , x_1, x_2, \dots, x_n where $\hat{v}_1 = \hat{x}(11.952\%)$, $\tilde{x} = \hat{x}(50\%)$ and $\hat{v}_3 = \hat{x}(88.048\%)$, then the estimates are

H-graph for normal and chisq(4)

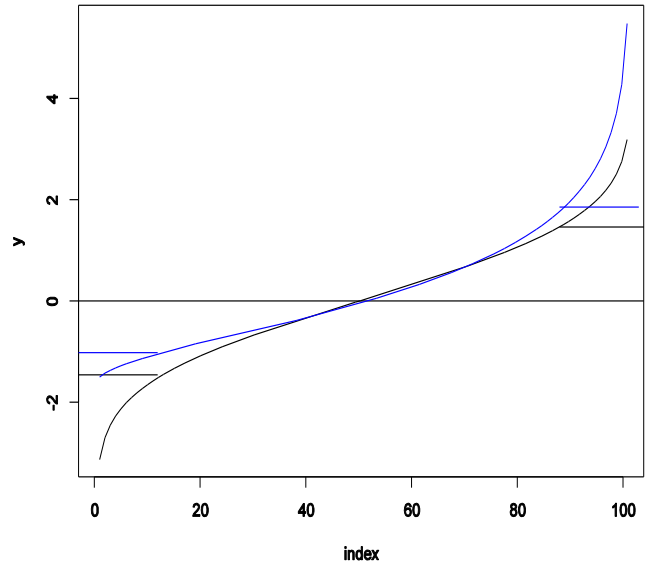


Figure 5. H-graph for chi-square (blue) and normal distributions and the normal has $H_1 = 0.25, H_2 = 0.25, H_3 = 0.25$ and $H_4 = 0.25$ while the chi-square has $H_1 = 0.15, H_2 = 0.20, H_3 = 0.29$ and $H_4 = 0.36$.

$$\hat{D} = \frac{1}{n} \sum_{i=1}^n |x_i - \tilde{x}|,$$

$$\hat{D}^+ = \frac{1}{n} \sum_{i=1}^n (x_i - \tilde{x}) I(x_i > \tilde{x}),$$

$$\hat{D}^- = \frac{1}{n} \sum_{i=1}^n (\tilde{x} - x_i) I(x_i < \tilde{x}),$$

$$\hat{D}^{\hat{v}_3} = \frac{1}{n} \sum_{i=1}^n (x_i - \tilde{x}) I(x_i > \hat{v}_3),$$

$$\hat{D}^{\hat{v}_1} = \frac{1}{n} \sum_{i=1}^n (\tilde{x} - x_i) I(x_i < \hat{v}_1),$$

$$\hat{D}^{(\hat{v}_1, \tilde{x})} = \frac{1}{n} \sum_{i=1}^n (\tilde{x} - x_i) I(\hat{v}_1 < x_i < \tilde{x})$$

$$\hat{D}^{(\tilde{x}, \hat{v}_3)} = \frac{1}{n} \sum_{i=1}^n (x_i - \tilde{x}) I(\tilde{x} < x_i < \hat{v}_3)$$

and

$$\hat{D}^{(\hat{v}_1, \hat{v}_3)} = \frac{1}{n} \sum_{i=1}^n |\tilde{x} - x_i| I(\hat{v}_1 < x_i < \hat{v}_3)$$

Also it is assumed that $x_i \neq \hat{v}, \hat{v}_1, \hat{v}_3$ and $i = 1, 2, \dots, n$. Hence,

$$\hat{S} = s = \frac{\hat{D}^+ - \hat{D}^-}{\hat{D}} = \hat{H}^+ - \hat{H}^- \quad (7)$$

and

$$\hat{K} = k = \frac{\hat{D}^{\hat{v}_1} - (\hat{D}^{(\hat{v}_1, \hat{v})} + \hat{D}^{(\hat{v}, \hat{v}_3)}) + \hat{D}^{\hat{v}_3}}{\hat{D}} = \hat{H}_1 -$$



$$(\hat{H}_2 + \hat{H}_3) + \hat{H}_4 = (\hat{H}_1 - \hat{H}_2) + (\hat{H}_4 - \hat{H}_3) = \hat{H}_1 - \hat{H}_{23} + \hat{H}_4(8)$$

The empirical mean and variances of these estimates from normal distribution are given in Table 3 using 10000 randomly generated normal samples for each sample size.

Table 3. The mean and variances of \hat{H}_1, \hat{H}_2, s and k from normal with replications of 10000

n	s		k		\hat{H}_1		\hat{H}_2	
	mea	Var	Me	var	me	var	me	Var
10	0.0	0.0	0.3	0.0	0.3	0.0	0.1	0.0
20	0.0	0.0	0.1	0.0	0.2	0.0	0.2	0.0
30	0.0	0.0	0.0	0.0	0.2	0.0	0.2	0.0
40	0.0	0.0	0.0	0.0	0.2	0.0	0.2	0.0
50	0.0	0.0	-	0.0	0.2	0.0	0.2	0.0
75	0.0	0.0	-	0.0	0.2	0.0	0.2	0.0
100	0.0	0.0	-	0.0	0.2	0.0	0.2	0.0

		006							
20	0.0	0.0	0.0	0.0	0.2	0.0	0.2	0.0	
0	005	044	004	010	50	003	50	005	
50	-	0.0	0.0	0.0	0.2	0.0	0.2	0.0	
0	0.0	018	002	004	50	001	50	002	
		002							
10	0.0	0.0	0.0	0.0	0.2	.00	0.2	0.0	
00	006	009	002	002	50	006	50	001	

From Table 3 the empirical variances of $\hat{H}_1 = h_1, \hat{H}_2 = h_2, \hat{H}_3 = h_3, \hat{H}_4 = h_4, s$ and k are

$$\widehat{var}(s) \approx \frac{0.90}{n}, \tag{9}$$

$$\widehat{var}(k) \approx \frac{0.20}{n}, \tag{10}$$

$$\widehat{var}(h_1) = \widehat{var}(h_4) \approx \frac{0.06}{n}, \tag{11}$$

and

$$\widehat{var}(h_2) = \widehat{var}(h_3) \approx \frac{0.10}{n} \tag{12}$$

These empirical variances are very good until for small sample sizes. Note that the mean and variances of h_3 and h_4 is omitted because they have the same results as h_2 and h_1 , respectively.

Figures 7 and 8 show the histogram and H-graph for simulated s and k using normal data and it is clear that the normal distribution gives a very good approximation to s and k until for small sample sizes such as 15 and 25.

H-graph

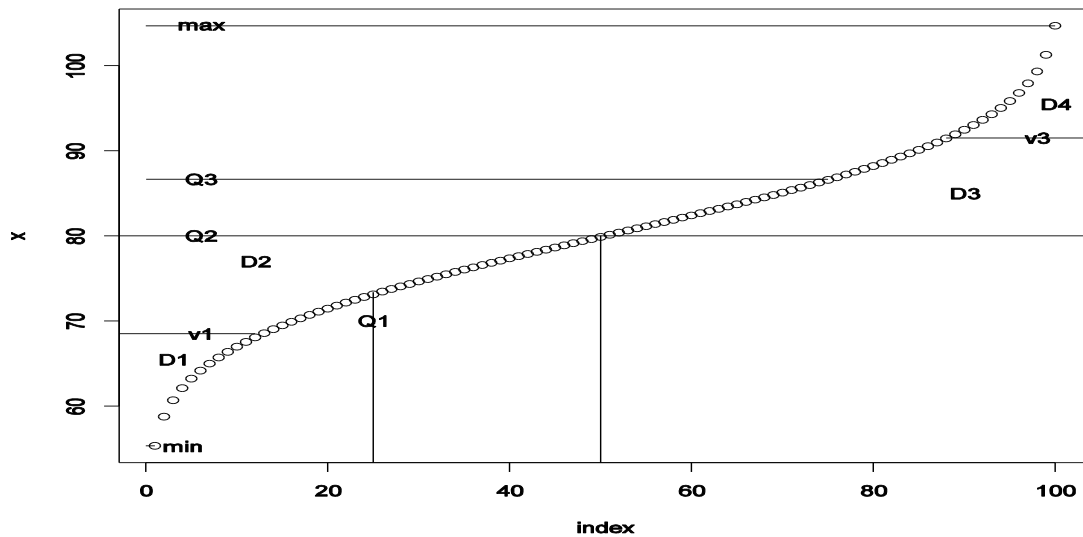


Figure 6. H-graph for the original data $N(80,10)$ and $n = 100$

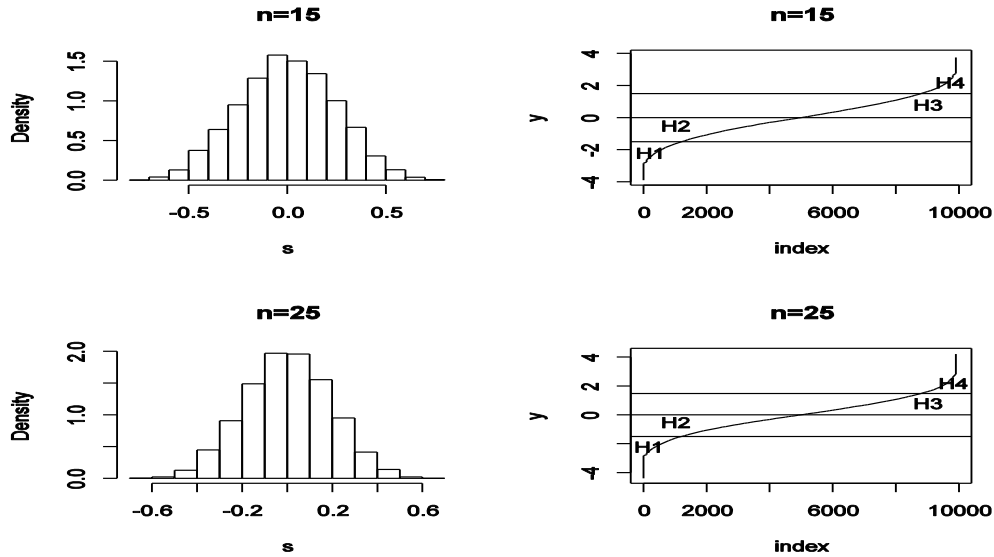


Figure 7. histogram and H-graph for s statistic using 10000 simulated standard normal data for different sample sizes where $H_1 = 0.248, H_2 = 0.252, H_3 = 0.256, H_4 = 0.244$ for $n = 15$ and $H_1 = 0.249, H_2 = 0.252, H_3 = 0.253, H_4 = 0.246$ for $n = 25$.

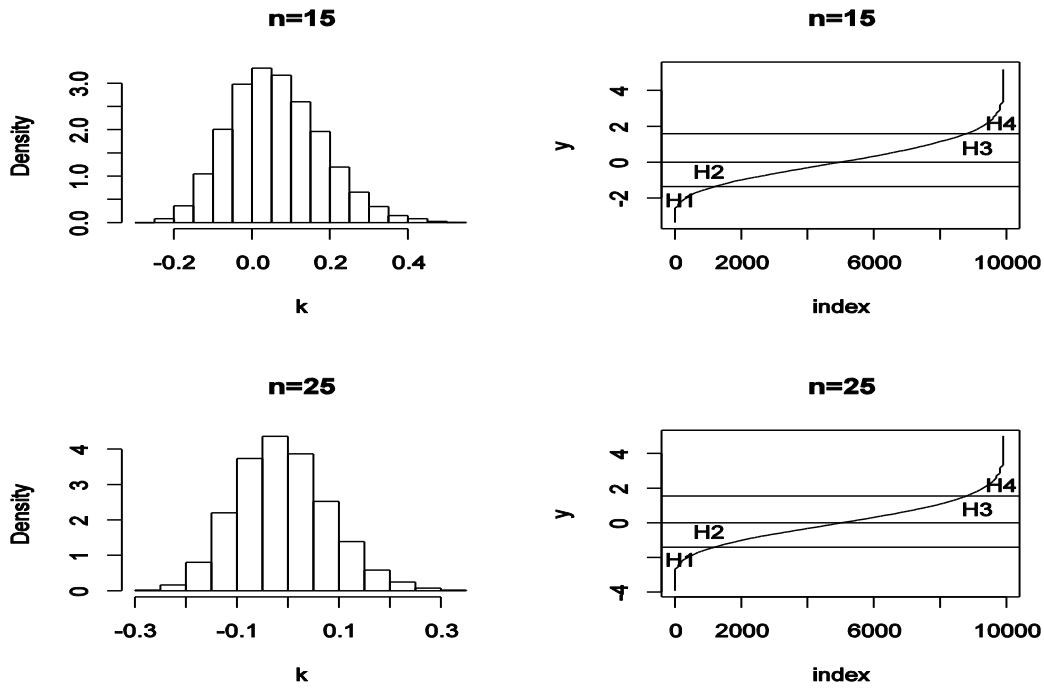


Figure 8. Histogram and H-graph for k statistic using 10000 simulated standard normal data for different sample sizes where $H_1 = 0.219, H_2 = 0.238, H_3 = 0.265, H_4 = 0.278$ for $n = 15$ and $H_1 = 0.228, H_2 = 0.245, H_3 = 0.253, H_4 = 0.274$ for $n = 25$.



5. MIDDLE-SIDES EQUALITY AND NORMALITY TESTS

A. Middle-sides equality or kurtosis test

In some applications it is important to test for kurtosis is zero (middle-side equality), leptokurtic (sides mass) and platykurtic (middle mass) with respect to the normal distribution. The null and alternative hypotheses can be written as

$$H_0: K = 0, \quad H_1: K \neq 0, \quad H_0: K \leq 0, \\ H_1: K > 0 \text{ and } H_0: K \geq 0, \\ H_1: K < 0$$

By applying the standard results, it can be shown that the statistic

$$z_k = \frac{k}{\sqrt{0.20/n}} \tag{13}$$

has an approximate standard normal distribution under the null hypothesis of normality. Reject $K = 0$ if $|z_k| > z_{1-\alpha/2}$. A one-sided test of tail inequality rejected if $z_k > z_{1-\alpha}$ and a one-sided test of peak inequality rejected if $z_k < z_\alpha$.

1) Power study

A good test satisfies a nominal Type I error (reject the null hypothesis when it is true) and large power (reject the null hypothesis when it is false). The statistic z_k is compared with Anscombe and Glynn (1983) test

$$z_\beta = \left[1 - 2/9c_1 - \left\{ (1 - 2/c_1) / (1 + c_3 \{ 2/(c_1 - 4) \}^2)^{1/3} \right\} \right] / (2/9c_1)^{1/2} \tag{14}$$

where

$$c_1 = 6 + (8/c_2) \{ 2/c_2 + (1 + 4/c_2)^{1/2} \}, \\ c_2 = \{ 6(n^2 - 5n + 2) / (n + 7)(n + 3) \} \{ 6(n + 3)(n + 5) / n(n - 2)(n - 3) \}^{1/2}, \\ c_3 = \{ b_2 - 3(n - 1) / (n + 1) \} / \{ 24n(n - 2)(n - 3) / (n + 1)^2(n + 3)(n + 5) \}^{1/2}$$

and Bonett and Seier (2002) test

$$z_w = (n + 2)^{1/2} (w - 3) / 3.54 \tag{15}$$

where $w = 13.29(\ln \hat{\sigma} - \ln d)$, $\hat{\sigma} = \sqrt{\sum(x - \bar{x})^2/n}$ and $d = \sum|x - \bar{x}|/n$.

For the empirical study the three tests are included and the following parameters are to be used $n = [25, 50, 100]$ based on repetitions 10000 and nominal type I error $\alpha = 0.05$ for one and two tailed test. All simulations were done in the software R, the source code of the programs is not listed here and it can be obtained from the author by request. The normal samples were generated in R with the function `rnorm()` and all random samples were generated independently from each other. For the calculation of the test statistic of z_β and z_w tests the already implemented functions `anscomb.test()` and `bonett.test()` in R (package `moments`) are used.

The z_β test and z_w are known to be a powerful tests; see, Bonett (2002). Tables 4 and 5 compare the empirical nominal type I error and the empirical power of the two-sided and one-sided z_β , z_w and z_k tests at $\alpha = 0.05$ for all distributions in set A and set B.

Table 4. Empirical type I error and power for two-tailed kurtosis tests and $\alpha = 0.05$

	n = 25			n = 50			n = 100		
	Z_β	Z_w	Z_k	Z_β	Z_w	Z_k	Z_β	Z_w	Z_k
Normal	0.052	0.049	0.049	0.054	0.050	0.050	0.052	0.050	0.050
Set A									
Beta(0.25,0.25)	93.1	90.5	97.2	*	99.9	100	*	100	100
Beta(0.5,0.5)	81.5	66.0	82.7	99.3	95.9	99.4	*	100	100
Uniform	44.1	28.7	40.1	88.5	62.4	71.3	99.9	93.5	96.1
Beta(1.5,1.5)	22.3	14.9	18.5	60.4	34.3	40.0	95.3	68.8	70.1
Logistic	13.0	12.2	10.2	19.6	18.9	14.7	32.8	32.1	24.6
Laplace	28.8	34.6	32.1	49.5	63.8	59.0	77.2	90.5	87.6
Set B									
gl(0,1,-.85,-.85)	88.1	91.3	91.1	100	100	100	100	100	100
gl(0,1,-.75,-.75)	84.8	88.5	87.1	98	99.3	99.0	99.0	100	100
gl(0,1,-.5,-.5)	69.4	73.6	69.7	92.1	95.1	93.2	99.6	99.9	99.8
gl(0,1,-.25,-.25)	44.0	45.5	40.0	70.1	73.9	67.3	91.8	94.4	91.0
gl(0,1,-.15,-.15)	30.4	30.8	25.1	51.7	54.0	44.3	76.7	80.3	70.1
gl(0,1,-.10,-.10)	24.1	23.6	19.0	41.1	43.0	33.2	65.9	67.6	55.9
gl(0,1,-.05,-.05)	18.1	17.4	14.2	30.5	30.2	22.8	50.6	51.0	40.0

*the program fails to give the results

Table 5. Empirical type I error and power for one-tailed kurtosis tests and $\alpha = 0.05$

	$n = 25$			$n = 50$			$n = 100$		
	Z_β	Z_w	Z_k	Z_β	Z_w	Z_k	Z_β	Z_w	Z_k
Normal	0.049	0.044	0.047	0.051	0.049	0.050	0.048	0.049	0.051
Set A									
Beta(0.25,0.25)	94.6	95.3	98.3	99.8	100	100	100	100	100
Beta(0.5,0.5)	87.7	79.0	89.6	99.7	98.3	99.7	100	100	100
Uniform	57.9	45.2	53.8	93.6	76.1	82.8	100	97.3	98.1
Beta(1.5,1.5)	34.5	28.2	31.9	72.9	51.3	54.1	98.1	82.5	82.0
Logistic	17.8	15.3	13.3	27.5	25.2	20.0	42.2	40.8	34.7
Laplace	38.6	42.2	40.1	61.7	72.5	69.0	84.3	94.1	93.1
Set B									
gl(0,1,-.85,-.85)	92.0	93.7	93.7	99.6	99.8	99.9	100	100	100
gl(0,1,-.75,-.75)	88.4	90.4	90.0	89.7	99.5	99.4	100	100	100
gl(0,1,-0.5,-0.5)	76.5	78.2	76.1	94.9	96.6	95.5	99.9	99.9	99.9
gl(0,1,-.25,-.25)	52.5	51.0	45.5	77.1	78.5	72.0	95.0	96.4	94.1
gl(0,1,-.15,-.15)	38.2	36.0	31.8	61.4	60.6	53.0	84.4	85.8	78.7
gl(0,1,-.10,-.10)	32.2	30.1	26.0	50.0	49.5	41.2	74.1	74.7	66.0
gl(0,1,-.05,-.05)	23.9	21.9	19.0	39.2	37.3	30.2	59.1	58.1	49.2

Tables 4 and 5 show the empirical one-tail and two-tailed type I error rate and power for tests. For type I error, the z_β test is slightly liberal for all sample sizes in two-tailed test and very close to nominal value for all samples sizes in one-tail test, z_w test is very close to nominal value for all sample size, one-tail test and two-tailed test except for $n = 25$ in one-tail test it is less than nominal value while z_k test is very close to nominal value for all sample sizes, one-tail test and two-tailed test except for $n = 25$ in one-tail test it is conservative.

For the power, Tables 4 and 5 show that the z_β has the most power in the distributions that have K in the range $(-0.16, 0.16)$ and competitive to z_k in the range $[-0.16, -1]$ while it is the weakest in the range $[0.16, 1]$, the z_w has the most power in the distributions which have K in the range $[0.16, 1]$ and competitive to z_β in the range $[-0.16, 0.16]$ while it is the weakest in the range $[-0.16, -1]$ and the z_k shows the most power in the distributions which have K in the range $[-0.16, -1]$ and very competitive to z_w in the range $[0.16, 1]$ while it is the weakest in the range $(-0.16, 0.16)$. Therefore it can conclude that the good test for kurtosis can be applied as

$$\begin{aligned} z_k & \text{ if } K \leq -0.16 \\ z_\beta & \text{ if } -0.16 < K < 0.16 \\ z_w & \text{ if } K \geq 0.16 \end{aligned}$$

B. Omnibus normality test

One of the most used distributions in statistical analysis is the normal distribution. Consequently, the development of tests for departures from normality became an important subject of statistical research. There are many approaches for normality test and the most famous approach consists of testing for normality using the third (β_1) and fourth (β_2) moments of observations x_1, \dots, x_n known as sample skewness $\sqrt{b_1}$ and sample kurtosis b_2 . Tests that can only detect deviations in either

the skewness or the kurtosis are called shape tests. The test that are able to cover both alternatives are called omnibus test. The probably most popular omnibus test is the Jarque-Bera test (1980) that is defined as

$$JB = \frac{n}{6} \left((\sqrt{b_1})^2 + \frac{(b_2 - 3)^2}{4} \right) \quad (16)$$

This is called JB statistic and has asymptotically χ^2_2 distributed; see, Jarque and Bera (1980, 1987), Thadewald and Buning (2007) and Gel and Gastwirt (2008).

The proposed omnibus normality test based on MAD_{med} is defined as

$$N_{sk} = N_1 = \frac{s^2}{\widehat{var}(s)} + \frac{k^2}{\widehat{var}(k)} = \frac{n s^2}{0.90} + \frac{n k^2}{0.20} \quad (17)$$

Under the null hypothesis and assuming that the two summands are independent then N_1 would be chi-squared (χ^2) distributed with two degrees of freedom. Figure 9 makes an attempt to show the correlation of s and k from several sample sizes. For all sample sizes there is no structure to recognize in the graph. Also, the convergence of the N_{sk} statistic to its asymptotic distribution is tried to be visualized in Figure 10. For each histogram in this figure, the N_{sk} statistic was calculated for $m = 10000$ realizations of standard normally generated random samples of the corresponding sample size n . Additionally, the theoretical probability distribution function of chi-squared distribution with 2 degrees of freedom is plotted in each histogram so that one is able to compare the goodness-of-fit of the empirical distribution with the theoretical distribution. For all sample sizes it is clear that the chi-squared with 2 degree of freedom gives a very good fit to statistic N_{sk} . This supports the assumption of independence between s and k .

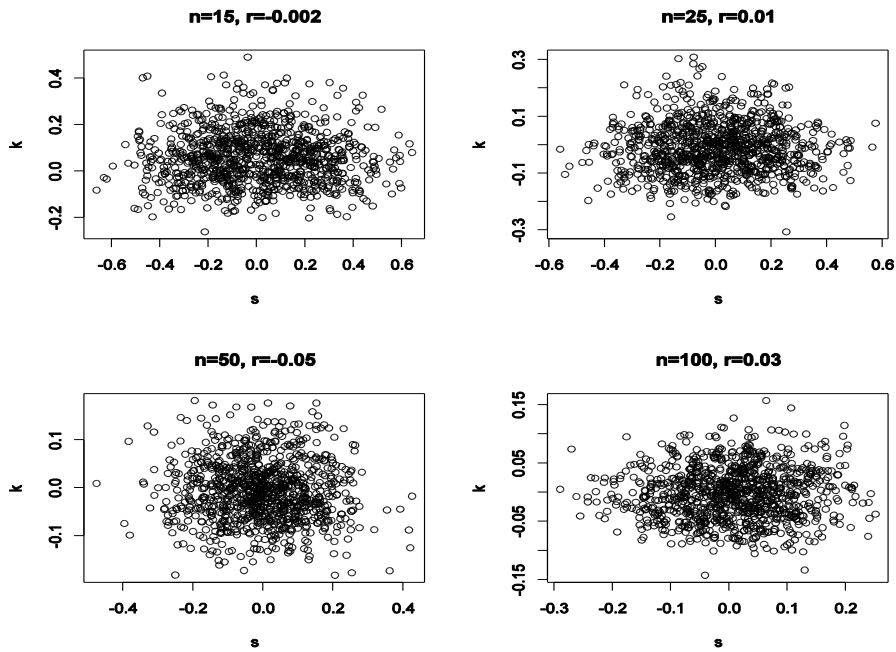


Figure 9. Scatter plot of s and k for 1000 randomly generated normal samples for each sample size.

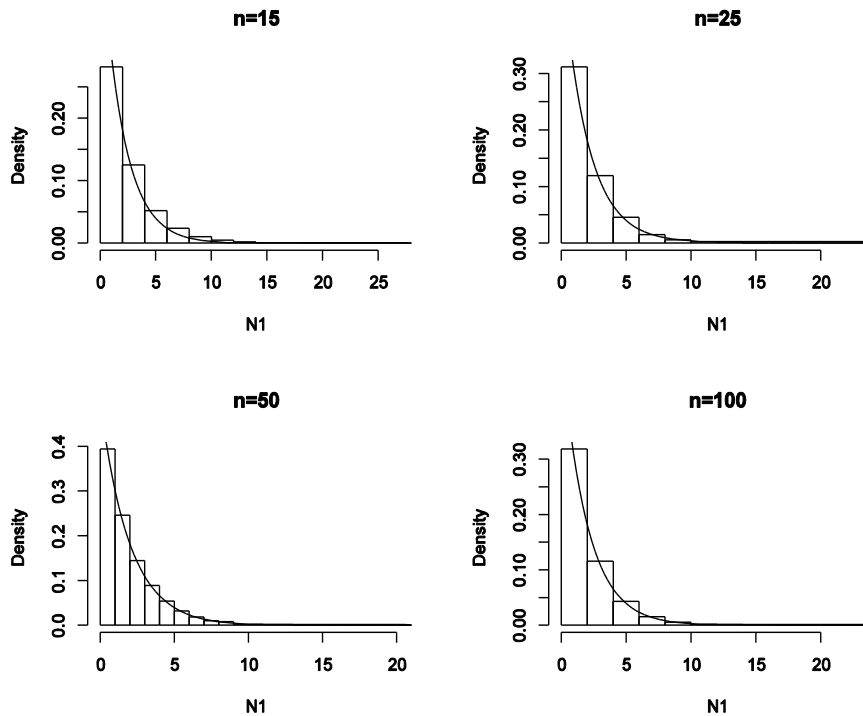


Figure 10. Histogram of the N_1 statistic for several sample sizes together with the pdf of the χ^2_2 distribution. For each sample size, 10000 standard normal samples were generated.

Table 6. Empirical type I error and power for normal tests using $\alpha = 0.05$

				$n = 25$				
Alternative	k	N_1	JB	Alternative	s	k	N_1	JB
Normal	0	0.049	0.031	Beta (1,0.5)	-0.333	-0.155	60.6	6.2
Beta(0.25,0.25)	-0.371	98.8	0.2	Beta (2,1)	-0.207	-0.092	24.7	3.5
Beta(0.5,0.5)	-0.264	78.0	0.1	Beta (3,2)	-0.086	-0.063	9.6	1.9
Uniform	-0.155	31.1	0	Chi-square (1)	0.645	0.141	86.9	84.2
Beta(1.5,1.5)	-0.107	15.4	0	Chi-square (2)	0.442	0.058	55.5	61.2
Logistic	0.056	9.1	12.0	Chi-square (4)	0.306	0.026	30.0	38.5
Laplace	0.165	27.1	27.1	lognormal (0,1)	0.575	0.215	81.0	83.8
gl(0,1,-.85,-.85)	0.653	90.5	87.6	lognormal (0,0.5)	0.306	0.059	31.4	43.6
gl(0,1,-.75,-.75)	0.568	85.9	83.4	Weibull (.5,1)	0.814	0.379	98.1	96.4
gl(0,1,-0.50,-0.50)	0.360	67.1	67.5	Weibull (1,1)	0.442	0.058	57.2	60.5
gl(0,1,-0.10,-0.10)	0.105	16.8	23.3	Weibull (2,1)	0.030	-0.020	5.1	1.9
				$n = 50$				
Normal	0	0.049	0.040	Beta (1,0.5)	-0.333	-0.155	90.1	27.1
Beta(0.25,0.25)	-0.371	100	64.1	Beta (2,1)	-0.207	-0.092	47.4	10.9
Beta(0.5,0.5)	-0.264	98.6	2.6	Beta (3,2)	-0.086	-0.063	15.9	1.0
Uniform	-0.155	61.4	0	Chi-square (1)	0.645	0.141	99.0	99.0
Beta(1.5,1.5)	-0.107	30.8	0	Chi-square (2)	0.442	0.058	85.8	95.1
Logistic	0.056	12.8	22.8	Chi-square (4)	0.306	0.026	52.6	77.1
Laplace	0.165	51.5	50.2	lognormal (0,1)	0.575	0.215	98.3	99.6
gl(0,1,-.85,-.85)	0.653	99.6	99.1	lognormal (0,0.5)	0.306	0.059	56.1	80.1
gl(0,1,-.75,-.75)	0.568	98.9	98.5	Weibull (.5,1)	0.814	0.379	100	100
gl(0,1,-0.5,-0.5)	0.360	92.0	92.4	Weibull (1,1)	0.442	0.058	85.0	95.3
gl(0,1,-0.10,-0.10)	0.105	28.3	43.4	Weibull (2,1)	0.030	-0.020	15.7	20.2
				$n = 100$				
Normal	0	0.051	0.044	Beta (1,0.5)	-0.333	-0.155	99.8	99.7
Beta(0.25,0.25)	-0.371	100	100	Beta (2,1)	-0.207	-0.092	80.1	74.4
Beta(0.5,0.5)	-0.264	100	100	Beta (3,2)	-0.086	-0.063	30.0	5.6
Uniform	-0.155	92.7	56.2	Chi-square (1)	0.645	0.141	100	100
Beta(1.5,1.5)	-0.107	60.1	9.0	Chi-square (2)	0.442	0.058	99.1	100
Logistic	0.056	20.1	37.0	Chi-square (4)	0.306	0.026	84.2	99.1
Laplace	0.165	83.8	78.6	lognormal (0,1)	0.575	0.215	100	100
gl(0,1,-.85,-.85)	0.653	100	100	lognormal (0,0.5)	0.306	0.059	84.3	98.9
gl(0,1,-.75,-.75)	0.568	100	100	Weibull (.5,1)	0.814	0.379	100	100
gl(0,1,-0.5,-0.5)	0.360	99.6	99.6	Weibull (1,1)	0.442	0.058	99.0	100
gl(0,1,-0.10,-0.10)	0.105	50.0	68.8	Weibull (2,1)	0.030	-0.020	28.1	49.8

1) Power study

The statistic N_{sk} is compared with the most popular and used moment test for normality the Jarque-Bera test (1980) that defined in equation (15). For the empirical study the two tests for normality are included and the following parameters are to be used $n = [25, 50, 100]$ based on repetitions 10000 and nominal type I error $\alpha = 0.05$. All simulations were done in the software R and the function of the test statistic JB is already implemented in R (package moments) `jarque.test()`.

Table 6 shows the results of simulation study for several symmetric and asymmetric distributions. For type I error, the empirical Type I error for JB test is quite less than nominal value for small sample sizes and conservative for large sample size while empirical Type I error for the

test N_{sk} is very close to nominal value for all used sample sizes. For the power, Table 6 shows that JB has the most power in the distributions that have kurtosis in the range $[-0.025, 0.36]$ while the statistic N_{sk} is the most power in the distributions that have k ranges $[-1, -0.025]$ and $(0.36, 1]$ regardless of the skewness value.

6. CONCLUSION

Two measures of shape were introduced with graphical display based on mean absolute deviation about median. The measure of skewness was based on the partitions of MAD_{med} into two parts to obtain zero for any symmetric distribution while the proposed measure of middle-sides equality was based on the partitions of MAD_{med} into four parts in terms of specific percentiles to get zero for normal distribution. The middle-sides equality measure



had clear meaning where the middle mass is compared with sides mass with respect to normal distribution.

Making a decision about goodness of fit for a data without looking at a graphic makes the investigation not complete. A famous and often cited quote of J.W. Tukey “there is no excuse for failing to plot and look”. Based on four partitions of MAD_{med} an informative graph was produced that could provide a more insight into the nature of the data and assess goodness of fit for a data set to a theoretical model to know if the assumption of a common distribution is justified. The H-graph enriched the visual information offered by the histogram and boxplot.

The tests for kurtosis (z_k) and normality (N_{sk}) were simple, easy to compute, did not require special tables of critical values where the chi-squared distribution with 2 degree of freedom is used and had a good power and Type I error control in comparisons with Anscombe-Glynn, Bonett-Seier and Jarque-Bera tests. With respect to kurtosis test, the statistic z_k was more powerful than z_β and z_w in platykurtic distributions and very competitive to z_w in leptokurtic distributions. With respect to normal test N_{sk} was more powerful than Jarque-Bera’ test in all kurtosis ranges of distributions except the

REFERENCES

- [1] Anscombe, F.J. and Glynn, W.J., 1983. Distribution of the kurtosis statistic b_2 for normal statistics. *Biometrika*, 70, 227–234.
- [2] Balanda, K., MacGillivray, H.L., 1988. Kurtosis: a critical review. *American Statistician*, 42, 111–119.
- [3] Bonett, D.G. and Seier E., 2002. A test of normality with high uniform power. *Computational Statistics & Data Analysis*, 40, 435–445.
- [4] DeCarlo, L.T., 1997. On the meaning and use of kurtosis. *Psychological Methods*, 3, 292–307
- [5] Filliben, J.J., 1975. The probability plot correlation coefficient test for normality. *Technometrics*, 17, 111–117.
- [6] Hodges, J.L., Lehmann, E.L., 1956. The efficiency of some nonparametric competitors of the t-test. *Annals of Mathematical Statistics*, 27, 224–235.
- [7] Hu, L.-T., Bentler, P.M., Kano, Y., 1990. Can test statistics in covariance structure models be trusted? *Psychology Bulletin*, 112, 351–362.
- [8] Jarque, C.M. and Bera, A.K., 1980. Efficient tests for normality, homoscedasticity and serial independence of regression residuals. *Economics Letters*, 6, 255–259.
- [9] Jarque, C.M. and Bera, A.K., 1987. A Test for normality of observations and regression residuals. *International Statistical Review*, 55, 163–172.
- [10] Gel, S. and Gastwirt, J., 2008. A robust modification of the Jarque-Bera test of normality. *Economic Letters*, 99, 30–32.
- [11] Groeneveld, R.A. and Meeden, G., 1984. Measuring skewness and kurtosis. *The Statistician*, 33, 391–399.
- [12] Micceri, T., 1989. The unicorn, the normal curve, and other improbable creatures. *Psychology Bulletin*, 105, 156–166
- [13] Oja, H., 1981. On location, scale, skewness and kurtosis of univariate distributions. *Scandinavian Journal of Statistics*, 8, 154–168.
- Pearson, K., 1905. Skew variation, a rejoinder. *Biometrika* 4, 169–212.
- [14] Perez, M.J. and Gomez, S., 1990. A characterization of the distribution function: The dispersion function. *Statistics and Probability Letters*, 10, 235–239.
- [15] Ramberg, J.S., Tadikamalla, P.R., Dudewicz, E.J., Mykytka, C.F., 1979. A probability distribution and its uses in fitting data. *Technometrics*, 21, 201–214.
- [16] Seier, E. and Bonett, D.G. 2011. A polyplot for visualizing location, spread, skewness and kurtosis. *The American Statistician*, 65, 303–313.
- [17] Thadewald, T. and H. Buning, H., 2007. Jarque-Bera test and its competitors for testing normality – A power comparison. *Journal of Applied Statistics*, 37, 87–105.
- [18] Thode, H.C., 2002. *Testing for normality*. Marcel Dekker, New York.
- [19] Tukey, J.W., 1977. *Exploratory data analysis*. Addison-Wesley.
- [20] van Zwet, W.R., 1964. *Convex transformations of random variables*. Mathematical Centre Tract 7. Mathematisch Centrum Amsterdam.