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Quintic spline collocation method for fractional boundary value problems



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Abstract The spline collocation method is a competent and highly effective mathematical tool for constructing the approximate solutions of boundary value problems arising in science, engineering and mathematical physics. In this paper, a quintic polynomial spline collocation method is employed for a class of fractional boundary value problems (FBVPs). The FBVPs are expressed in terms of Caputo's fractional derivative in this approach. The consistency relations are derived in order to compute the approximate solutions of FBVPs. Finally, numerical results are given, which demonstrate the effectiveness of the numerical scheme.

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1. Introduction

The idea behind fractional derivatives (and integrals) is no more new now. Fractional derivative gives a perfect aid to characterize the memory and hereditary properties of various processes and materials, therefore differential equations of fractional order are being used in modeling of electrical and mechanical properties of various real materials, rock's rheological properties, and in many other areas. The most of the differential equations of fractional order do not have analytical solutions, this is the main reason for finding new numerical methods for the solutions of fractional differential equations becomes a hot topic for the research community. An extensive research has been carried out to obtain the numerical schemes which are numerically stable for both linear and nonlinear differential equations of fractional order.

Many authors used the spline technique to establish the accurate and efficient numerical schemes for the solution of boundary value problems. For example, Siddiqi and Akram constructed many numerical schemes with the help of different spline functions such as polynomial splines and non-polynomial splines for the solution of various BVPs (Siddiqi et al., 2007; Siddiqi and Akram, 2008, 2007). Also, Akram and Aslam (2016) established the Adomian decomposition method (ADM) and the reproducing kernel method (RKM) for the solution of fourth order three-point boundary value problem. The theory of FBVPs has received considerable interest in recent years. The interest towards the theory of existence and uniqueness of solutions to FBVPs is apparent from the recent publications (Ahmad and Nieto, 2009; Bai, 2010; Zhang, 2006). FBVPs occur in the explanation of many physical stochastic-transport processes and in the inspection of liquid filtration which arises in a strongly porous's medium (Taukenova and Shkhanukov-Lafishev, 2006). Also, boundary value problems with integral boundary conditions establish a very fascinating and predominant class of problems. Two, three, four, multi point and nonlocal boundary value problems

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are the special cases of such problems. Cellular systems and population dynamics are some phenomena in which boundary conditions of integral type occur (Chen et al., 2010). Analysis and representation of many physical systems demand solutions of fractional boundary value problems. Recently, a great amount of effort has been employed in attempting to find stable and robust numerical and analytical methods in order to solve FBVPs (Al-Rabtah et al., 2012; Akram and Tariq, 2015; Almeida and Bastos, 2015; Doha et al., 2011; Edwards et al., 2002; Ford et al., 2014; Jin et al., 2015, 2016; Kumar et al., 2015a,b; Nouri and Siavashani, 2014; Rehmana and Khan, 2012; Secer et al., 2013; Seifollahi and Shamloo, 2013).

The fourth order boundary value problem explains an elastic-bending-beams's static deflection through a nonlinear loading (Gupta, 1988). It also depicts a prototype-equation's steady state for phase transitions in systems of condensed matter (Grinstein and Luther, 1976) and is helpful in understanding traveling waves in a suspension bridge (Lazer and McKenna, 1990). In this paper, consider the following fourth order boundary value problems for linear fractional differential equations:

$$y^{(4)}(x) + D^\alpha p(x)y(x) = g(x), \quad x \in [a, b], \quad (1)$$

subject to

$$y(a) - A_1 = y(b) - A_2 = y''(a) - B_1 = y''(b) - B_2 = 0, \quad (2)$$

where $A_i, B_i, i = 1, 2$ are real constants. The functions $p(x)$ and $g(x)$ are continuous on the interval $[a, b]$ and D^α denotes fractional derivative in Caputo's sense. In general, the analytical solution of Eqs. (1) and (2) cannot be obtained for arbitrary choices of $p(x)$ and $g(x)$. The problem Eq. (1) arise in the plate deflection theory. When $\alpha = 0$, Eq. (1) is reduced to the classical fourth order boundary value problem.

In this study, Caputo's fractional derivative is used. This operator is widely applied in modeling of the material's mechanical properties, modeling of the viscoelastic behavior, signal processing, diffusion problems, bioengineering and mathematical finance models etc. Also in this study, we focus on providing a numerical scheme, based on quintic polynomial spline collocation method, to solve fourth order boundary value problems for linear fractional differential equations.

Quintic polynomial spline scheme is commonly used in order to solve differential equations. If the solutions of FBVPs are needed at various locations in the given region then the spline solutions guarantee to give the information of spline interpolation between mesh points. Also the strong advantage of this scheme is to provide smooth continuous approximations to exact solutions at every point of the range of integration.

The paper is organized as follows: some preliminaries of fractional calculus are given in Section 2. In Section 3, quintic polynomial spline method is developed for the solution of fractional differential equation. The matrix form of the proposed scheme is discussed in Section 4. In Section 5, numerical results are given to compare and illustrate the efficiency of the method.

2. Preliminaries

There are several definitions to the generalization of the notion of fractional differentiation. Riemann-Liouville and Caputo's

are most common definitions. But Caputo's approach is suitable for real world physical problems because it defines integer order initial conditions for fractional differential equations. The Riemann- Liouville left and right fractional integral of order $\alpha > 0$ is defined as

$$I_{a+}^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} y(s) ds, \quad x > 0$$

and

$$I_{b-}^\alpha y(x) = -\frac{1}{\Gamma(\alpha)} \int_b^x (x-s)^{\alpha-1} y(s) ds,$$

respectively. Also, Caputo's fractional derivative of order α is defined as

$$D^\alpha y(x) = I^{m-\alpha} D^m y(x),$$

where D^m is ordinary differential operator.

If $\alpha > 0, m-1 \leq \alpha < m, \delta > -1, m \in \mathbb{N}, \lambda, \mu \in \mathbb{R}$ and $y(x)$ is continuous function, then the following results hold:

$$D^\alpha C = 0, \quad C \text{ is constant}$$

$$D^\alpha(\lambda y(x) + \mu q(x)) = \lambda D^\alpha y(x) + \mu D^\alpha q(x)$$

$$I^\alpha x^\delta = \frac{\Gamma(\delta+1)}{\Gamma(\delta+1+\alpha)} x^{\delta+\alpha}$$

Theorem 1. Let $0 < \alpha < 1$ and assume that f and g are analytic on $(a-h, a+h)$. Then

$$D_a^\alpha [fg](x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} g(a)(f(x) - f(a)) + (D_a^\alpha g(x))f(x) + \sum_{k=1}^{\infty} \binom{\alpha}{k} (I_a^{k-\alpha} g(x)) D_a^k f(x).$$

For more properties of fractional derivatives, we refer to (Diethelm, 2010; Kilbas et al., 2006; Lakshmikantham and Vatsala, 2008; Ortigueira and Machado, 2015; Ortigueira and Trujillo, 2012; Podlubny, 1999).

3. Quintic spline functions

Let $x_i = ih$ ($i = 0, 1, \dots, n, h = \frac{b-a}{n}, n > 0$) be grid points of the uniform partition of $[a, b]$ into the subintervals $[x_{i-1}, x_i]$. Let $y(x)$ be the exact solution of Eq. (1) and S_i be an approximation to $y_i = y(x_i)$ obtained by the spline function $T_i(x)$ passing through the points (x_i, S_i) and (x_{i+1}, S_{i+1}) . Consider that each quintic polynomial spline segment $T_i(x)$ has the following form:

$$T_i(x) = a_i(x - x_{i-1})^5 + b_i(x - x_{i-1})^4 + c_i(x - x_{i-1})^3 + d_i(x - x_{i-1})^2 + e_i(x - x_{i-1}) + f_i, \quad (3)$$

$i = 1, 2, \dots, n$, along with the requirement that $T_i(x) \in C^4[a, b]$ and

$$S(x) = T_i(x) \quad \forall x \in [x_{i-1}, x_i], \quad i = 1, 2, \dots, n. \quad (4)$$

In order to develop the consistency relations between the values of spline and its derivatives at knots, let

$$T_i(x_{i-1}) = S_{i-1}, \quad T_i(x_i) = S_i,$$

$$T_i^{(2)}(x_{i-1}) = M_{i-1}, \quad T_i^{(2)}(x_i) = M_i,$$

$$T_i^{(4)}(x_{i-1}) = F_{i-1}, \quad T_i^{(4)}(x_i) = F_i.$$

It is to be noticed that the spline can be written in terms of S_i s and any three derivatives at the boundaries of each subinterval. To define spline in terms of S_i s, M_i s, and F_i s, the coefficients introduced in Eq. (3) are calculated, as

$$a_i = \frac{1}{120h}(F_i - F_{i-1}), \quad b_i = \frac{1}{24}F_{i-1},$$

$$c_i = \frac{1}{6h}(M_i - M_{i-1}) - \frac{h}{36}(F_i + 2F_{i-1}),$$

$$d_i = \frac{1}{2}M_{i-1}, e_i = \frac{1}{h}(S_i - S_{i-1}) - \frac{h}{6}(M_i + 2M_{i-1}) + \frac{h^3}{360}(7F_i + 8F_{i-1}), \quad f_i = S_{i-1}.$$

Applying the first and third derivative continuities at the knots, i.e. $T_i^\rho(x_i) = T_{i+1}^\rho(x_i)$, $\rho = 1$ and 3 , the following useful relations are obtained, as

$$M_{i+1} + 4M_i + M_{i-1} = \frac{6}{h^2}(S_{i+1} - 2S_i + S_{i-1}) + \frac{h^2}{60}(7F_{i+1} + 16F_i + 7F_{i-1}), \quad (5)$$

$$M_{i+1} - 2M_i + M_{i-1} = \frac{h^2}{6}(F_{i+1} + 4F_i + F_{i-1}). \quad (6)$$

Using Eqs. (5) and (6), the following consistency relation in terms of the fourth derivative of spline F_i and S_i , $i = 1, 2, \dots, n$, is derived, as

$$S_{i+2} - 4S_{i+1} + 6S_i - 4S_{i-1} + S_{i-2} = \frac{h^4}{120}(F_{i+2} + 26F_{i+1} + 66F_i + 26F_{i-1} + F_{i-2}) \quad i = 2, 3, \dots, n - 2, \quad (7)$$

where

$$F_i = [-D_{x_{i-1}}^2 p(x)T(x) + g(x)]|_{x=x_i}. \quad (8)$$

Since the system (7) gives $(n - 3)$ linear algebraic equations in the $(n - 1)$ unknowns (S_i , $i = 1, 2, \dots, n - 1$), therefore two more equations (end conditions) are required. The two end conditions can be obtained using Taylor series and the method of undetermined coefficients. Two end equations are

$$-2S_0 + 5S_1 - 4S_2 + S_3 = -h^2M_0 + \frac{h^4}{120}(18F_0 + 65F_1 + 26F_2 + F_3) \quad (9)$$

and

$$S_{n-3} - 4S_{n-2} + 5S_{n-1} - 2S_n = -h^2M_n + \frac{h^4}{120}(F_{n-3} + 26F_{n-2} + 65F_{n-1} + 18F_n). \quad (10)$$

Suppose that F_0 is linearly approximated between F_1 and F_2 as, $F_0 = 2F_1 - F_2$ and also F_n is linearly approximated between F_{n-1} and F_{n-2} as, $F_n = 2F_{n-1} - F_{n-2}$.

For $i = 1$, the consistency relation can be taken, as

$$-2S_0 + 5S_1 - 4S_2 + S_3 = -h^2M_0 + \frac{h^4}{120}(101F_1 + 8F_2 + F_3). \quad (11)$$

For $i = 2$, the consistency relation can be written, as

$$S_0 - 4S_1 + 6S_2 - 4S_3 + S_4 = \frac{h^4}{120}(28F_1 + 65F_2 + 26F_3 + F_4). \quad (12)$$

For $i = n - 2$, the consistency relation can be taken, as

$$S_n - 4S_{n-1} + 6S_{n-2} - 4S_{n-3} + S_{n-4} = \frac{h^4}{120}(F_{n-4} + 26F_{n-3} + 65F_{n-2} + 28F_{n-1}). \quad (13)$$

For $i = n - 1$, the consistency relation can be written, as

$$S_{n-3} - 4S_{n-2} + 5S_{n-1} - 2S_n = -h^2M_n + \frac{h^4}{120}(F_{n-3} + 8F_{n-2} + 101F_{n-1}). \quad (14)$$

Lemma 1. Let $y \in C^6[a, b]$ then the local truncation errors t_i , $i = 1, 2, \dots, n - 1$ associated with the Eqs. (11), (12), (7), (13) and (14) are

$$t_i = \begin{cases} \frac{13}{180}h^6y^{(6)}(x_1) + O(h^7), & i = 1, \\ \frac{113}{360}h^6y^{(6)}(x_1) + O(h^7), & i = 2, \\ \frac{-1}{12}h^6y^{(6)}(x_i) + O(h^7), & i = 3, 4, \dots, n - 3, \\ \frac{113}{360}h^6y^{(6)}(x_{n-1}) + O(h^7), & i = n - 2, \\ \frac{13}{180}h^6y^{(6)}(x_{n-1}) + O(h^7), & i = n - 1. \end{cases}$$

Proof. In order to obtain the local truncation errors t_i , $i = 1, 2, \dots, n - 1$, firstly rewrite Eqs. (11), (12), (7), (13) and (14) in the following form, as

$$t_1 = -2y_0 + 5y_1 - 4y_2 + y_3 + h^2M_0 - \frac{h^4}{120}[101y_1^{(iv)} + 8y_2^{(iv)} + y_3^{(iv)}],$$

$$t_2 = y_0 - 4y_1 + 6y_2 - 4y_3 + y_4 - \frac{h^4}{120}[28y_1^{(iv)} + 65y_2^{(iv)} + 26y_3^{(iv)} + y_4^{(iv)}],$$

$$t_i = y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2} - \frac{h^4}{120}[y_{i+2}^{(iv)} + 26y_{i+1}^{(iv)} + 66y_i^{(iv)} + 26y_{i-1}^{(iv)} + y_{i-2}^{(iv)}],$$

$$t_{n-2} = y_n - 4y_{n-1} + 6y_{n-2} - 4y_{n-3} + y_{n-4} - \frac{h^4}{120}[y_{n-4}^{(iv)} + 26y_{n-3}^{(iv)} + 65y_{n-2}^{(iv)} + 28y_{n-1}^{(iv)}]$$

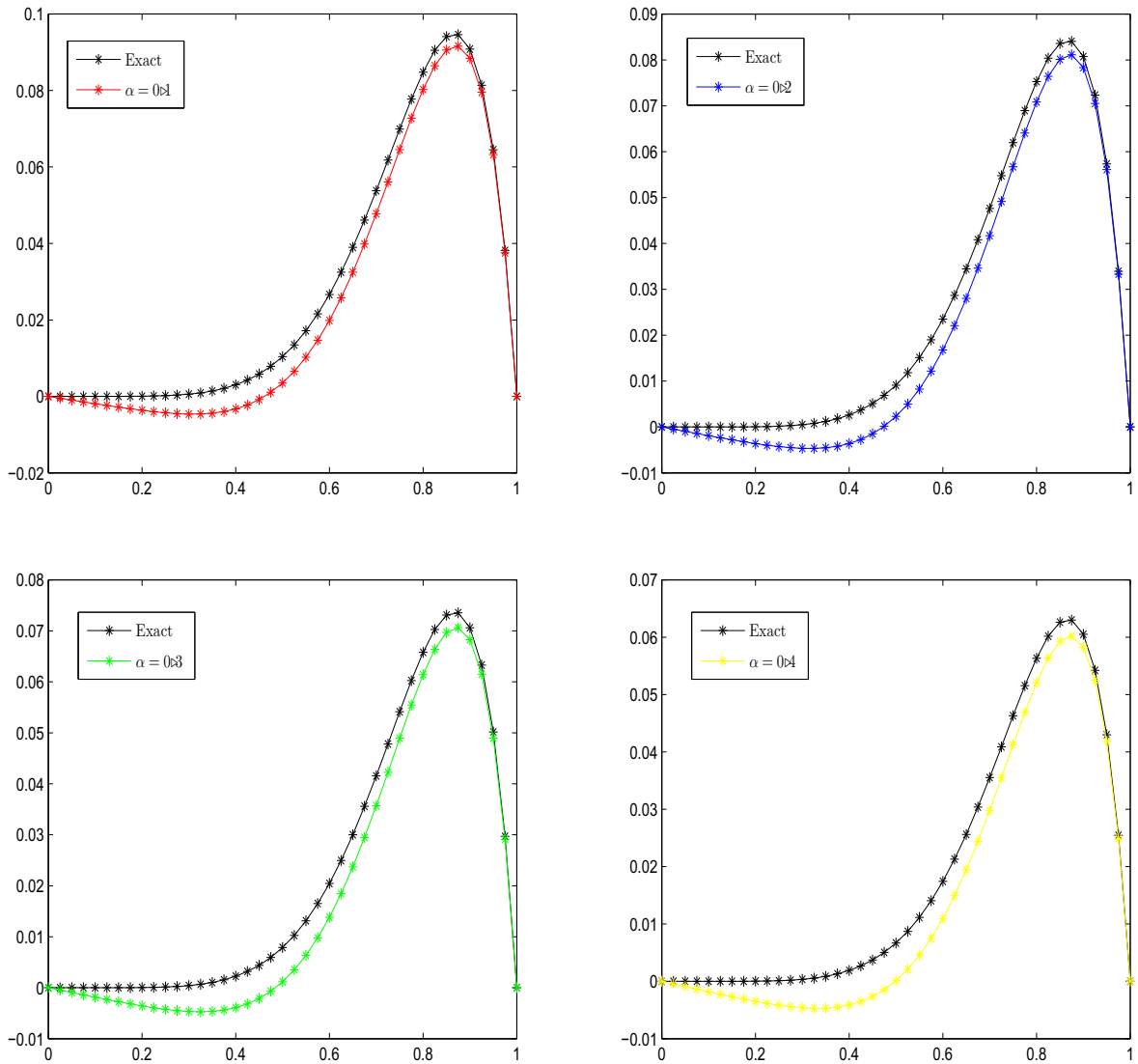
and

$$t_{n-1} = y_{n-3} - 4y_{n-2} + 5y_{n-1} - 2y_n + h^2M_n - \frac{h^4}{120}[y_{n-3}^{(iv)} + 8y_{n-2}^{(iv)} + 101y_{n-1}^{(iv)}].$$

The terms y_0 , y_1 , $y_1^{(iv)}$, y_2 , $y_2^{(iv)}$, etc are expanded about the point x_1 using Taylor's series and the expressions for

Table 2 Maximum absolute errors and order of convergence (O. C.) for Example 2.

n	$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.3$	
	Error	O.C.	Error	O.C.	Error	O.C.
10	$8.44E-002$		$7.61E-002$		$6.68E-002$	
20	$2.37E-002$	1.8324	$2.19E-002$	1.7970	$2.01E-002$	1.7318
40	$6.90E-003$	1.7802	$6.90E-003$	1.6663	$6.80E-003$	1.5636

**Figure 2** Exact and approximate solutions of Example 2 with different values of α .

Then

$$\|E\|_{\infty} \leq \frac{\|C^{-1}\|_{\infty} \|T\|_{\infty}}{1 - h^4 \|C^{-1}\|_{\infty} \|D\|_{\infty} \|M^{-1}\|_{\infty} \|N\|_{\infty}} \cong O(h^2). \quad (29)$$

Proof. From Lemma 2,

$$\begin{aligned} \|E\|_{\infty} &= \max_{1 \leq i \leq n-1} |e_i| \\ &\leq \frac{\|C^{-1}\|_{\infty} \|T\|_{\infty}}{1 - h^4 \|C^{-1}\|_{\infty} \|D\|_{\infty} \|M^{-1}\|_{\infty} \|N\|_{\infty}}, \end{aligned} \quad (30)$$

provided that $h^4 \|C^{-1}\|_\infty \|D\|_\infty \|M^{-1}\|_\infty \|N\|_\infty < 1$. From Siddiqi and Akram (2008),

$$\|C^{-1}\|_\infty = \frac{1}{384h^4} (b-a)^4 \left(1 + \frac{8h^3}{(b-a)^3} \right).$$

Also,

$$\begin{aligned} \|D\|_\infty &= 1 \text{ and } \|N\|_\infty \\ &= \frac{h^{-\alpha}}{\Gamma(6-\alpha)} (240 - 288\alpha + 113\alpha^2 - 18\alpha^3 + \alpha^4). \end{aligned}$$

By substituting the values of $\|C^{-1}\|_\infty$, $\|D\|_\infty$, $\|M^{-1}\|_\infty$ and $\|N\|_\infty$ in Eq. (30) and using Eq. (23), it can be written as

$$\|E\|_\infty \leq \frac{h^2 c_3 Z_6 \xi (720\Gamma(6-\alpha) - h^{4-\alpha} \lambda_1) ((b-a)^3 + 8h^3)}{(720\Gamma(6-\alpha) - h^{4-\alpha} \lambda_1) - 720\Gamma(6-\alpha) \xi ((b-a)^3 + 8h^3) \lambda_2 h^{-\alpha}} \cong O(h^2), \tag{31}$$

where $Z_6 = \max_{a \leq x \leq b} |y^6(x)|$. \square

Theorem 2. Let $y(x)$ be the exact solution of the FBVP Eq. (1) with boundary condition Eq. (2) and y_i , $i = 0, 1, 2, \dots, n-1$, satisfy the discrete BVP Eq. (22). Moreover, if $e_i = y_i - S_i$, then

$$\|E\|_\infty = O(h^2).$$

6. Numerical results

In this section, to check the accuracy, efficiency and validity of the method, some examples of suggested method are given.

Example 1. Consider the following FBVP:

$$y^{(4)}(x) + 0.05D^\alpha y(x) = g(x), \quad x \in [0, 1],$$

Appendix A

$$\begin{aligned} D_{x_{i-1}}^\alpha p(x) T_i(x) |_{x=x_i} &= \frac{h^{-\alpha}}{\Gamma(1-\alpha)} S_{i-1} (p(x_i) - p(x_{i-1})) + p(x_i) \left(\frac{2h^{4-\alpha}(-2+\alpha)\alpha}{720\Gamma(4-\alpha)} F_{i-2} + \frac{h^{4-\alpha}(-4+\alpha)\alpha(361+\alpha(-226+25\alpha))}{720\Gamma(6-\alpha)} F_{i-1} \right. \\ &+ \frac{2h^{4-\alpha}\alpha(-778+\alpha(605+\alpha(-137+10\alpha)))}{720\Gamma(6-\alpha)} F_i + \frac{(-5+\alpha)^2(-4+\alpha)h^{4-\alpha}\alpha}{720\Gamma(6-\alpha)} F_{i+1} \\ &+ \frac{-240(-2+\alpha)(-5+\alpha)(-4+\alpha)\alpha h^{-\alpha}}{720\Gamma(6-\alpha)} S_{i-2} + \frac{-360h^{-\alpha}(12+(-9+\alpha)\alpha)(-5+\alpha)(-4+\alpha)}{720\Gamma(6-\alpha)} S_{i-1} \\ &+ \left. \frac{h^{-\alpha}720(6+(-6+\alpha)\alpha)(-5+\alpha)(-4+\alpha)}{\Gamma(6-\alpha)} S_i + \frac{-120(-5+\alpha)^2(-4+\alpha)\alpha h^{-\alpha}}{720\Gamma(6-\alpha)} S_{i+1} \right) \\ &+ \sum_{k=1}^{\infty} \binom{\alpha}{k} \left(\frac{h^{4+k-\alpha}}{\Gamma(5+k-\alpha)} F_{i-1} + \frac{h^{4+k-\alpha}}{\Gamma(6+k-\alpha)} (F_i - F_{i-1}) + \frac{h^{4+k-\alpha}}{\Gamma(1+k-\alpha)} S_{i-1} \right. \\ &+ \left. \frac{(6(h^{4+k-\alpha}(F_{i-2} - 37F_{i-1} - 23F_i - F_{i+1}) - 120(S_{i-2} - 3S_{i-1} + 3S_i - S_{i+1})))}{720\Gamma(4+k-\alpha)} \right. \\ &- \left. \frac{(6(h^{4+k-\alpha}(F_{i-2} + 4F_{i-1} + F_i) - 120(S_{i-2} - 2S_{i-1} + S_i)))}{720\Gamma(3+k-\alpha)} - \frac{1}{720\Gamma(2+k-\alpha)} (h^{4+k-\alpha}(2F_{i-2} + 25F_{i-1} + 20F_i + F_{i+1}) \right. \\ &\left. - 120(2S_{i-2} + 3S_{i-1} - 6S_i + S_{i+1})) \right) D_{x_{i-1}}^k p(x). \end{aligned}$$

with

$$y(0) = 0, \quad y(1) = 0,$$

$$y''(0) = 0, \quad y''(1) = 8.$$

The exact solution of this problem is $x^5 - x^4$. The present scheme is applied with different values of α and results are shown in Table 1 and Fig. 1.

Furthermore in the limit, as α goes to zero, the method provides a solution for the integer order system. From numerical results, it is observed that suggested scheme is of $O(h^2)$.

Example 2. Consider the following FBVP:

$$y^{(4)}(x) + D^\alpha x y(x) = g(x), \quad x \in [0, 1],$$

with

$$y(0) = 0, \quad y(1) = 0,$$

$$y''(0) = 0, \quad y''(1) = 26(\alpha - 1).$$

The exact solution of this problem is $x^{6+\alpha} - x^{8-\alpha}$. The present scheme is applied with different values of α and results are shown in Table 2 and Fig. 2.

7. Conclusion

Collocation method is established for the approximate solution of fractional differential equation along with boundary conditions, using quintic spline. The suggested method also utilizes the properties of fractional derivatives in order to solve this problem. This numerical scheme is computationally captivating. Descriptive examples show applications of this problem. It is proved that the method is of $O(h^2)$.

$$\begin{aligned} \mu_1 &= \frac{2h^{4-\alpha}(-2+\alpha)\alpha}{720\Gamma(4-\alpha)}(p(x))|_{x=x_i} + \sum_{k=1}^{\infty} \binom{\alpha}{k} \left(\frac{h^{4+k-\alpha}}{120\Gamma(4+k-\alpha)} - \frac{h^{4+k-\alpha}}{120\Gamma(3+k-\alpha)} - \frac{h^{4+k-\alpha}}{360\Gamma(2+k-\alpha)} \right) D_{x_{i-1}}^k p(x)|_{x=x_i}, \\ \mu_2 &= \frac{h^{4-\alpha}(-4+\alpha)\alpha(361+\alpha(-226+25\alpha))}{720\Gamma(6-\alpha)} p(x)|_{x=x_i} + \sum_{k=1}^{\infty} \binom{\alpha}{k} \left(\frac{h^{4+k-\alpha}}{\Gamma(5+k-\alpha)} - \frac{h^{4+k-\alpha}}{\Gamma(6+k-\alpha)} - \frac{37h^{4+k-\alpha}}{120\Gamma(4+k-\alpha)} - \frac{h^{4+k-\alpha}}{30\Gamma(3+k-\alpha)} \right. \\ &\quad \left. + \frac{5h^{4+k-\alpha}}{144\Gamma(2+k-\alpha)} \right) D_{x_{i-1}}^k p(x)|_{x=x_i}, \\ \mu_3 &= \frac{2h^{4-\alpha}\alpha(-778+\alpha(605+\alpha(-137+10\alpha)))}{720\Gamma(6-\alpha)} p(x)|_{x=x_i} + \sum_{k=1}^{\infty} \binom{\alpha}{k} \left(\frac{h^{4+k-\alpha}}{\Gamma(6+k-\alpha)} - \frac{23h^{4+k-\alpha}}{120\Gamma(4+k-\alpha)} - \frac{h^{4+k-\alpha}}{120\Gamma(3+k-\alpha)} + \frac{h^{4+k-\alpha}}{36\Gamma(2+k-\alpha)} \right) \\ &\quad \times D_{x_{i-1}}^k p(x)|_{x=x_i}, \\ \mu_4 &= \frac{(-5+\alpha)^2(-4+\alpha)h^{4-\alpha}\alpha}{720\Gamma(6-\alpha)} p(x)|_{x=x_i} + \sum_{k=1}^{\infty} \binom{\alpha}{k} \left(-6\frac{h^{4+k-\alpha}}{\Gamma(4+k-\alpha)} + \frac{h^{4+k-\alpha}}{720\Gamma(2+k-\alpha)} \right) D_{x_{i-1}}^k p(x)|_{x=x_i}, \\ \eta_1 &= \frac{-240(-2+\alpha)(-5+\alpha)(-4+\alpha)\alpha h^{-\alpha}}{720\Gamma(6-\alpha)} p(x)|_{x=x_i} + \sum_{k=1}^{\infty} \binom{\alpha}{k} \left(\frac{-720h^{4+k-\alpha}}{\Gamma(4+k-\alpha)} + \frac{h^{4+k-\alpha}}{\Gamma(3+k-\alpha)} - \frac{h^{4+k-\alpha}}{3\Gamma(2+k-\alpha)} \right) D_{x_{i-1}}^k p(x)|_{x=x_i}, \\ \eta_2 &= \frac{-360h^{-\alpha}(12+(-9+\alpha)\alpha)(-5+\alpha)(-4+\alpha)}{720\Gamma(6-\alpha)} p(x)|_{x=x_i} + \frac{h^{-\alpha}}{\Gamma(1-\alpha)}(p(x_i) - p(x_{i-1})) \\ &\quad + \sum_{k=1}^{\infty} \binom{\alpha}{k} \left(\frac{2160h^{4+k-\alpha}}{\Gamma(4+k-\alpha)} - \frac{2h^{4+k-\alpha}}{\Gamma(3+k-\alpha)} - \frac{h^{4+k-\alpha}}{2\Gamma(2+k-\alpha)} + \frac{h^{4+k-\alpha}}{\Gamma(1+k-\alpha)} \right) D_{x_{i-1}}^k p(x)|_{x=x_i}, \\ \eta_3 &= \frac{h^{-\alpha}720(6+(-6+\alpha)\alpha)(-5+\alpha)(-4+\alpha)}{720\Gamma(6-\alpha)} p(x)|_{x=x_i} + \sum_{k=1}^{\infty} \binom{\alpha}{k} \left(\frac{-2160h^{4+k-\alpha}}{\Gamma(4+k-\alpha)} + \frac{h^{4+k-\alpha}}{\Gamma(3+k-\alpha)} + \frac{h^{4+k-\alpha}}{\Gamma(2+k-\alpha)} \right) D_{x_{i-1}}^k p(x)|_{x=x_i}, \\ \eta_4 &= \frac{-120(-5+\alpha)^2(-4+\alpha)\alpha h^{-\alpha}}{720\Gamma(6-\alpha)} p(x)|_{x=x_i} + \sum_{k=1}^{\infty} \binom{\alpha}{k} \left(\frac{720h^{4+k-\alpha}}{\Gamma(4+k-\alpha)} - \frac{h^{4+k-\alpha}}{6\Gamma(2+k-\alpha)} \right) D_{x_{i-1}}^k p(x)|_{x=x_i} \end{aligned}$$

and

$$g_i = g(x_i).$$

$$\begin{aligned} \mu_{11} &= h^{4-\alpha} \left(\frac{2}{\Gamma(5-\alpha)} - \frac{1}{\Gamma(6-\alpha)} + \frac{13}{180\Gamma(2-\alpha)} - \frac{53}{60\Gamma(4-\alpha)} \right) p(x)|_{x=x_1} \\ &\quad + \sum_{k=1}^{\infty} \binom{\alpha}{k} h^{4+k-\alpha} \left(\frac{2}{\Gamma(5+k-\alpha)} - \frac{1}{\Gamma(6+k-\alpha)} + \frac{13}{180\Gamma(2+k-\alpha)} - \frac{53}{60\Gamma(4+k-\alpha)} \right) D_{x_0}^k p(x)|_{x=x_1}, \\ \mu_{12} &= h^{4-\alpha} \left(\frac{-1}{\Gamma(5-\alpha)} + \frac{1}{\Gamma(6-\alpha)} - \frac{4}{180\Gamma(2-\alpha)} + \frac{20}{60\Gamma(4-\alpha)} \right) p(x)|_{x=x_1} \\ &\quad + \sum_{k=1}^{\infty} \binom{\alpha}{k} h^{4+k-\alpha} \left(\frac{-1}{\Gamma(5+k-\alpha)} + \frac{1}{\Gamma(6+k-\alpha)} - \frac{4}{180\Gamma(2+k-\alpha)} + \frac{20}{60\Gamma(4+k-\alpha)} \right) D_{x_0}^k p(x)|_{x=x_1}, \\ \eta_{10} &= \frac{h^{-\alpha}}{\Gamma(1-\alpha)}(p(x_1) - p(x_0)) + h^{-\alpha} \left(\frac{-7}{6\Gamma(2-\alpha)} - \frac{1}{\Gamma(4-\alpha)} \right) p(x)|_{x=x_1} + \sum_{k=1}^{\infty} \binom{\alpha}{k} h^{k-\alpha} \left(\frac{-7}{6\Gamma(2+k-\alpha)} - \frac{1}{\Gamma(4+k-\alpha)} \right) D_{x_0}^k p(x)|_{x=x_1}, \\ \eta_{11} &= h^{-\alpha} \left(\frac{2}{45\Gamma(2-\alpha)} - \frac{1}{30\Gamma(4-\alpha)} \right) p(x)|_{x=x_1} + \sum_{k=1}^{\infty} \binom{\alpha}{k} h^{k-\alpha} \left(\frac{2}{45\Gamma(2+k-\alpha)} - \frac{1}{30\Gamma(4+k-\alpha)} \right) D_{x_0}^k p(x)|_{x=x_1}, \\ \eta_{12} &= h^{-\alpha} \left(\frac{1}{180\Gamma(2-\alpha)} - \frac{1}{60\Gamma(4-\alpha)} \right) p(x)|_{x=x_1} + \sum_{k=1}^{\infty} \binom{\alpha}{k} h^{k-\alpha} \left(\frac{1}{180\Gamma(2+k-\alpha)} - \frac{1}{60\Gamma(4+k-\alpha)} \right) D_{x_0}^k p(x)|_{x=x_1}, \\ \theta &= \left(\frac{-1}{\Gamma(4-\alpha)} + \frac{1}{\Gamma(3-\alpha)} - \frac{1}{3\Gamma(2-\alpha)} \right) h^{2-\alpha} p(x)|_{x=x_1} + \sum_{k=1}^{\infty} \binom{\alpha}{k} \left(\frac{-1}{\Gamma(4+k-\alpha)} + \frac{1}{\Gamma(3+k-\alpha)} - \frac{1}{3\Gamma(2+k-\alpha)} \right) h^{2+k-\alpha} D_{x_0}^k p(x)|_{x=x_1}. \end{aligned}$$

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