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Stability and ultimate boundedness of solutions of some third order differential equations with delay

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Abstract This paper is devoted to study the boundedness, ultimate boundedness, and the asymptotic stability of solutions for a certain class of third-order nonlinear differential equations using Lyapunov's second method. Our results improve and form a complement to some earlier results in the literature.

1. Introduction

The investigation of the qualitative properties of third order differential equations (with and without delay) have been intensively discussed and are still being investigated in the literature. By employing the Lyapunov's method, many good and interesting results have been obtained concerning the boundedness, ultimate boundedness and the asymptotic stability of solutions for certain nonlinear differential equations. See, the papers of Ademola and Arawomo (2013); Ademola et al. (2013); Burton (2005); Hara (1971); Bao and Cao (2009); Pan and Cao (2010, 2011, 2012); Omeike (2010); Oudjedi et al. (2014); Remili and Beldjerd (2014); Remili and Oudjedi (2014); Remili and Damerdji Oudjedi (2014); Li and Lizhi (1987); Tunç (2007a, b, 2010); Yoshizawa (1966) and their references.

In 1992, (Zhu, 1992), established some sufficient conditions to ensure the stability, boundedness and ultimate boundedness of the solutions of the following third order non-linear delay differential equation

$$x''' + ax'' + bx' + f(x(t-r)) = e(t).$$

Recently, (Graef et al., 2015), studied the following third order non autonomous differential equation with delay

$$[g(x(t))x'(t)]'' + (h(x(t))x'(t))' + \varphi(x(t))x'(t) + f(x(t-r)) = e(t),$$

which is more general than those considered by Zhu (1992). Simulated by the above reasons, we investigate the boundedness, ultimate boundedness, and the asymptotic stability of solutions for a kind of third-order differential equation with delay as follows

$$[g(x''(t))x''(t)]' + (h(x'(t))x'(t))' + (\varphi(x(t))x(t))' + f(x(t-r)) = e(t), \quad (1.1)$$

where $r > 0$ is a fixed delay and $e,\ f,\ g,\ h$, and φ are continuous functions in their respective arguments with $f(0) = 0$. The continuity of functions $e,\ f,\ g,\ h$, and φ guarantees the existence of the solution of Eq. (1.1). In addition, it is also supposed that the derivatives $f'(x),\ \ g'(u),\ h'(y)$ and $\varphi'(x)$ exist and are continuous.

The main purpose of this paper is to establish criteria for the uniform asymptotic stability and, uniform ultimate boundedness, of solutions for the third order non-linear differential Eqs. (1.1). The results obtained in this investigation provide

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a good supplement to the existing results on the third order nonlinear delay differential equations in the literature as (Zhu, 1992; Graef et al., 2015).

The remainder of this paper is organized as follows. In Section 2, we give a theorem, which deals with asymptotic stability of the zero solution of the delay differential Eq. (1.1) with $e(t) = 0$. In Section 3, we introduced theorem which discusses the uniform boundedness, and uniform ultimate boundedness of the solutions of Eq. (1.1) for the case $e(t) \neq 0$. Eventually, some conclusions are given in Section 4.

2. Stability

Take general nonlinear non-autonomous delay differential equation in the form

$$x' = f(x_t), \quad x_t(\theta) = x(t+\theta), \quad -r \leqslant \theta \leqslant 0, \ t \geqslant 0, \tag{2.1}$$

where $f : C_H \to \mathbb{R}^n$ is a continuous mapping, $f(0) = 0$, $C_H := \{\phi \in (C[-r, 0], \ \mathbb{R}^n) : \|\phi\| \leqslant H\}$, and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(\phi)| < L(H_1)$ when $\|\phi\| < H_1$.

Lemma 2.1 Krasovskii, 1963. *If there is a continuous functional $V(t, \phi) : [0, +\infty) \times C_H \to [0, +\infty)$ locally Lipschitz in ϕ and wedges W_i such that*:

(i) *If $W_1(\|\phi\|) \leqslant V(t, \phi)$, $V(t, 0) = 0$ and $V'_{(2.1)}(t, \phi) \leqslant 0$. Then, the zero solution of (2.1) is stable. If in addition $V(t, \phi) \leqslant W_2(\|\phi\|)$ Then, the zero solution of (2.1) is uniformly stable.*

(ii) *If $W_1(\|\phi\|) \leqslant V(t, \phi) \leqslant W_2(\|\phi\|)$ and $V'_{(2.1)}(t, \phi) \leqslant -W_3(\|\phi\|)$. Then, the zero solution*of (2.1) is uniformly asymptotically stable.

Now, suppose that there are positive constants $g_0, g_1, h_0, h_1, \varphi_0, \varphi_1, \delta_0, \delta_1$ and μ_1 such that the following conditions which will be used on the functions that appeared in Eq. (1.1) are satisfied:

(i) $0 < g_0 \leqslant g(u) \leqslant g_1, \quad 0 < h_0 \leqslant h(y) \leqslant h_1,$
$0 < \varphi_0 \leqslant \varphi(x) \leqslant \varphi_1,$

(ii) $f(0) = 0, \frac{f(x)}{x} \geqslant \delta_0 > 0 \ (x \neq 0)$, and $|f'(x)| \leqslant \delta_1$ for all x,

(iii) $\frac{\delta_1}{\varphi_0} < \mu_1 < \frac{h_0}{g_1}$,

(iv) $\int_{-\infty}^{+\infty} (|g'(u)| + |h'(u)| + |\varphi'(u)|)du < \infty$.

For ease of exposition throughout this paper we will adopt the following notations

$$P(t) = g(x''(t)), \quad \theta_1(t) = \frac{P'(t)}{P^2(t)},$$
$$\theta_2(t) = h'(x'(t))x''(t) \text{ and } \theta_3(t) = \varphi'(x(t))x'(t). \tag{2.2}$$

and

$$\sigma_1(t) = \min\{x''(0), x''(t)\}, \quad \sigma_2(t) = \max\{x''(0), x''(t)\}, \tag{2.3}$$
$$\rho_1(t) = \min\{x(0), x(t)\}, \quad \rho_2(t) = \max\{x(0), x(t)\},$$
$$\psi_1(t) = \min\{x'(0), x'(t)\}, \quad \psi_2(t) = \max\{x'(0), x'(t)\}.$$

For the case $e(t) \equiv 0$, The stability result of this paper is the following theorem.

Theorem 2.2. *If in addition to the hypotheses (i)–(iv), suppose that the following is also satisfied*

$$r < \min\left\{\frac{2g_0(h_0 - \mu_1 g_1)}{g_1^2 \delta_1}, \frac{2g_0(\mu_1\varphi_0 - \delta_1)}{\delta_1(2\mu_1 g_0 + 1)}\right\},$$

Then every solution of (1.1) is uniformly asymptotically stable.

Proof. Eq. (1.1) is equivalent to the following system

$$\begin{aligned} x' &= y \\ y' &= \frac{z}{P(t)} \\ z' &= -\frac{h(y)}{P(t)}z - \theta_2(t)y - \theta_3(t)x - \varphi(x)y - f(x) \\ &\quad + \int_{t-r}^{t} y(s)f'(x(s))ds, \end{aligned} \tag{2.4}$$

The main tool in the proofs of our results is the continuously differentiable functional $W = W(t, x_t, y_t, z_t)$ defined as

$$W(t, x_t, y_t, z_t) = e^{-\frac{\omega(t)}{\mu}} V_1(t, x_t, y_t, z_t) = e^{-\frac{\omega(t)}{\mu}} V_1, \tag{2.5}$$

where

$$V_1 = \mu_1 F(x) + f(x)y + \frac{\varphi(x)}{2}y^2 + \frac{1}{2P(t)}z^2 + \mu_1 yz$$
$$+\frac{1}{2}\mu_1 h(y)y^2 + \lambda \int_{-r}^{0} \int_{t+s}^{t} y^2(\xi)d\xi ds, \tag{2.6}$$

$$\omega(t) = \int_0^t Q(s)ds, \quad \text{and} \quad Q(t) = |\theta_1(t)| + |\theta_2(t)| + |\theta_3(t)|, \tag{2.7}$$

such that $F(x) = \int_0^x f(u)du$ and $\theta_1, \ \theta_2, \ \theta_3$, are defined as (2.2). μ and λ are some positive constants which will be specified later in the proof. We observe that the above functional V_1 can be rewritten as follows

$$V_1 = \mu_1 F(x) + \frac{\varphi(x)}{2}\left(y + \frac{f(x)}{\varphi(x)}\right)^2 - \frac{f^2(x)}{2\varphi(x)} + \frac{1}{2P(t)}(z + \mu_1 P(t)y)^2$$
$$+ \frac{\mu_1(h(y) - \mu_1 P(t))}{2}y^2 + \lambda \int_{-r}^{0} \int_{t+s}^{t} y^2(\xi)d\xi ds.$$

Considering the conditions (i) and (iii), we derive that

$$\frac{\mu_1(h(y) - \mu_1 P(t))}{2} \geqslant \frac{\mu_1(h_0 - \mu_1 g_1)}{2} > 0.$$

It follows that there exists sufficiently small positive constant δ_2 such that

$$\frac{1}{2P(t)}(z + \mu_1 P(t)y)^2 + \frac{\mu_1(h(y) - \mu_1 P(t))}{2}y^2$$
$$\geqslant \delta_2 y^2 + \delta_2 z^2. \tag{2.8}$$

Under the hypotheses (i)–(iii), we have

$$\begin{aligned} \mu_1 F(x) - \tfrac{f^2(x)}{2\varphi(x)} &\geqslant \mu_1 \int_0^x \left(1 - \tfrac{f'(u)}{\mu_1 \varphi(x)}\right) f(u)du \\ &\geqslant \mu_1 \int_0^x \left(1 - \tfrac{\delta_1}{\mu_1 \varphi_0}\right) f(u)du \\ &\geqslant \delta_3 F(x), \end{aligned}$$

where

$$\delta_3 = \mu_1\left(1 - \frac{\delta_1}{\mu_1\varphi_0}\right) > \mu_1\left(1 - \frac{\mu_1}{\mu_1}\right) = 0.$$

Moreover, assumption (ii) implies

$$\mu_1 F(x) - \frac{f^2(x)}{2\varphi(x)} \geqslant \frac{\delta_3\delta_0}{2}x^2. \tag{2.9}$$

Clearly, from (2.9), (2.8) and (2.6), we have the following estimate

$$V_1 \geqslant \delta_2 y^2 + \delta_2 z^2 + \frac{\delta_3\delta_0}{2}x^2 + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds. \tag{2.10}$$

Since the integral $\int_{t+s}^{t} y^2(\xi)d\xi$ is positive, we can find a positive constant k_0, small enough such that the last inequality gives

$$V_1 \geqslant k_0(x^2+y^2+z^2), \tag{2.11}$$

where $k_0 = min\{\delta_2; \frac{\delta_3\delta_0}{2}\}$. Observe that

$$P'(t) = g'(x''(t))x'''(t).$$

Thus, from hypotheses (i) and (iv), we have

$$\begin{aligned}\omega(t) &= \int_0^t Q(s)ds \\ &\leqslant \frac{1}{g_0^2}\int_{\sigma_1(t)}^{\sigma_2(t)}|g'(u)|du + \int_{\rho_1(t)}^{\rho_2(t)}|\varphi'(u)|du + \int_{\psi_1(t)}^{\psi_2(t)}|h'(u)|du \\ &\leqslant \frac{1}{g_0^2}\int_{-\infty}^{+\infty}|g'(u)|du + \int_{-\infty}^{+\infty}(|\varphi'(u)|+|h'(u)|)du \leqslant N < \infty,\end{aligned}$$

where $\sigma_1,\ \sigma_2,\ \rho_1,\ \rho_2,\ \psi_1,\ \psi_2$ are defined as (2.3).

Therefore we can find a continuous function $W_1(|\Phi(0)|)$ with

$$W_1(|\Phi(0)|) \geqslant 0 \quad \text{and} \quad W_1(|\Phi(0)|) \leqslant W(t,\Phi).$$

The existence of a continuous function $W_2(\|\phi\|)$ which satisfies the inequality $W(t,\phi) \leqslant W_2(\|\phi\|)$, is easily verified.

Now, the time derivative of the functional $V_1(t,x_t,y_t,z_t)$, with respect to the system (2.4) can be calculated as follows

$$\begin{aligned}V'_{1_{(2.4)}} &= [f'(x) - \mu_1\varphi(x)]y^2 + \left[\frac{\mu_1 P(t) - h(y)}{P^2(t)}\right]z^2 + \Theta(t) + \lambda r y^2 \\ &\quad + \left(\frac{1}{P(t)}z + \mu_1 y\right)\int_{t-r}^{t} y(s)f'(x(s))ds - \lambda\int_{t-r}^{t} y^2(\xi)d\xi,\end{aligned}$$

where

$$\begin{aligned}\Theta(t) &= -\frac{1}{2}\theta_1(t)z^2 - \theta_2(t)\left(\frac{h(y)}{P^2(t)}yz + \frac{\mu_1}{2}y^2\right) \\ &\quad + \theta_3(t)\left(\frac{1}{2}y^2 - \frac{1}{P(t)}xz - \mu_1 xy\right) \\ &\leqslant \frac{1}{2}|\theta_1(t)|z^2 + |\theta_2(t)|\left(\frac{h_1}{g_0^2}|yz| + \frac{\mu_1}{2}y^2\right) \\ &\quad + |\theta_3(t)|\left(\frac{1}{2}y^2 + \frac{1}{g_0}|xz| + \mu_1|xy|\right).\end{aligned}$$

Using the Schwartz inequality $|uv| \leqslant \frac{1}{2}(u^2+v^2)$, we obtain

$$\begin{aligned}\Theta(t) &\leqslant \frac{1}{2}\left[|\theta_1(t)| + \left(\mu_1 + \frac{h_1}{g_0^2}\right)|\theta_2(t)|\right](y^2+z^2) \\ &\quad + \frac{1}{2}\left(1 + \mu_1 + \frac{1}{g_0}\right)|\theta_3(t)|(x^2+y^2+z^2) \\ &\leqslant \frac{k_1}{k_0}Q(t)V_1,\end{aligned}$$

where

$$k_1 = \frac{1}{2}max\left\{1 + \mu_1 + \frac{h_1}{g_0^2},\ 1 + \mu_1 + \frac{1}{g_0}\right\}.$$

Furthermore, from hypotheses (i) and (ii), we get

$$\begin{aligned}V'_{1_{(2.4)}} &\leqslant -[\mu_1\varphi_0 - \delta_1 - \lambda r]y^2 - \left[\frac{h_0 - \mu_1 g_1}{g_1^2}\right]z^2 + \frac{k_1}{k_0}Q(t)V_1 \\ &\quad + \left(\frac{z}{P(t)} + \mu_1 y\right)\int_{t-r}^{t} y(s)f'(x(s))ds - \lambda\int_{t-r}^{t} y^2(\xi)d\xi,\end{aligned}$$

Using again the Schwartz inequality, and making use of the fact that $|f'(x)| \leqslant \delta_1$, we obtain the following inequalities

$$\mu_1 y\int_{t-r}^{t} y(s)f'(x(s))ds \leqslant \frac{\delta_1\mu_1 r}{2}y^2 + \frac{\mu_1\delta_1}{2}\int_{t-r}^{t} y^2(\xi)d\xi, \tag{2.12}$$

and

$$\frac{1}{P(t)}z\int_{t-r}^{t} y(s)f'(x(s))ds \leqslant \frac{\delta_1 r}{2g_0}z^2 + \frac{\delta_1}{2g_0}\int_{t-r}^{t} y^2(\xi)d\xi.$$

After some rearrangement we obtain

$$\begin{aligned}V'_{1_{(2.4)}} &\leqslant -\left[\mu_1\varphi_0 - \delta_1 - \left(\lambda + \frac{\mu_1\delta_1}{2}\right)r\right]y^2 \\ &\quad - \left[\frac{h_0 - \mu_1 g_1}{g_1^2} - \frac{\delta_1 r}{2g_0}\right]z^2 + \frac{k_1}{k_0}Q(t)V_1 \\ &\quad + \left[\frac{\delta_1}{2}\left(\mu_1 + \frac{1}{g_0}\right) - \lambda\right]\int_{t-r}^{t} y^2(\xi)d\xi.\end{aligned} \tag{2.13}$$

Let

$$\begin{aligned}\lambda &= \frac{\delta_1}{2}\left(\mu_1 + \frac{1}{g_0}\right), \\ M_1 &= \mu_1\varphi_0 - \delta_1 - \frac{\delta_1}{2}\left(2\mu_1 + \frac{1}{g_0}\right)r, \\ M_2 &= \frac{h_0 - \mu_1 g_1}{g_1^2} - \frac{\delta_1 r}{2g_0}.\end{aligned}$$

Hence, the last inequality becomes

$$V'_{1_{(2.4)}} \leqslant -M_1 y^2 - M_2 z^2 + \frac{k_1}{k_0}Q(t)V_1.$$

Now, in view of the inequalities (2.11) and (2.5) and taking $\mu = \frac{k_0}{k_1}$ we obtain

$$\begin{aligned}W'_{(2.4)} &= e^{-\frac{k_1\omega(t)}{k_0}}\left(V'_{1_{(2.4)}} - \frac{k_1}{k_0}Q(t)V_1\right) \\ &\leqslant e^{-\frac{k_1\omega(t)}{k_0}}\left[-M_1 y^2 - M_2 z^2\right].\end{aligned}$$

Provided that

$$r < \min\left\{\frac{2g_0(h_0 - \mu_1 g_1)}{g_1^2\delta_1}, \frac{2g_0(\mu_1\varphi_0 - \delta_1)}{\delta_1(2\mu_1 g_0 + 1)}\right\},$$

one can conclude for some positive constant $D > 0$ that

$$W'_{(2.4)}(t,x_t,y_t,z_t) \leqslant -\ D(y^2+z^2),$$

where

$$D = e^{-\frac{k_1 N}{k_0}}\min\{M_1,\ M_2\}.$$

From (2.4), $W_3(\|X\|) = D(y^2+z^2)$ is positive definite function. Hence, Lemma 2.1 guarantees that the trivial solution of Eq. (1.1) is uniformly asymptotically stable and completes the proof of the Theorem. □

3. Boundedness of solutions

Consider

$$x' = f(t, x_t), x_t(\theta) = x(t+\theta), -r \leqslant \theta \leq 0 \tag{3.1}$$

where $f : \mathbb{R} \times C \to \mathbb{R}^n$, is continuous mappings and takes bounded sets into bounded sets.

Lemma 3.1 Burton, 1985. *Let $V(t,\varphi) : \mathbb{R} \times C \to \mathbb{R}$ be continuous and locally Lipschitz in φ. If*

$$(i)\quad W_0(|x(t)|) \leqslant V(t,x_t) \leqslant W_1(|x(t)|) + W_2\left(\int_{t-r}^{t} W_3(|x(t)|)ds\right),$$

$$(ii)\quad V'_{(3.1)} \leqslant -W_3(|x(t)|) + M,$$

for some $M > 0$, where $W_i (i = 0, 1, 2, 3)$ are wedges, then the solutions of (3.1) are uniformly bounded and uniformly ultimately bounded for bound B.

Now, we shall state and prove our main result on the boundedness and ultimate boundedness of (1.1) with $e(t) \neq 0$.

Theorem 3.2. *If hypotheses (i)–(iv) hold true, and in addition the following conditions are satisfied*

(j) $|e(t)| \leqslant m$, *for some* $m > 0$,
(jj) $\delta_0 > \frac{1+\varphi_1}{2}$,
(jjj) $\mu_2 = min\left\{\frac{1}{g_1}, \frac{2(h_0\varphi_0 - \delta_1 g_1)}{g_0(2+\varphi_1)}, \frac{(\mu_1\varphi_0 - \delta_1)}{h_1}\right\}$.

Then every solution of (1.1) is uniformly bounded and uniformly ultimately bounded provided r satisfies

$$r < \min\left\{\frac{2\delta_0 - (1+\varphi_1)}{\delta_1}, \frac{g_0(2(h_0 - \mu_1 g_1) - \mu_2 g_1^2)}{2g_1^2\delta_1}, \frac{2g_0(\mu_1\varphi_0 - \delta_1 - \mu_2 h_1)}{\delta_1(2h_1 + (2\mu_1 + \mu_2)g_0 + 2)}\right\}, \tag{3.2}$$

Proof. In the case $e(t) \neq 0$ The Eq. (1.1) is equivalent to the following system

$$\begin{aligned} x' &= y \\ y' &= \frac{z}{P(t)} \\ z' &= -\frac{h(y)}{P(t)}z - \theta_2(t)y - \theta_3(t)x - \varphi(x)y - f(x) \\ &\quad + \int_{t-r}^{t} y(s)f'(x(s))ds + e(t). \end{aligned} \tag{3.3}$$

As in Theorem 2.2, the proof of this theorem also depends on the differentiable Lyapunov functional U defined as

$$U(t, x_t, y_t, z_t) = e^{-\eta(t)}V(t, x_t, y_t, z_t) = e^{-\eta(t)}V, \tag{3.4}$$

Such that $V(t,x_t,y_t,z_t) = V_1(t,x_t,y_t,z_t) + V_2(t,x_t,y_t,z_t)$ where V_1 is defined as (2.6) and

$$V_2 = \frac{h(y)}{P(t)}F(x) + \frac{\mu_2}{2}x^2 + f(x)y + \frac{\varphi(x)}{2}y^2 + \mu_2 x(z + h(y)y) + \frac{1}{2P(t)}(z + h(y)y)^2,$$

where

$$\eta(t) = \int_0^t \left[\frac{1}{\omega}Q(s) + \frac{1}{\alpha}|\theta_4(t)|\right]ds,$$

where $Q(t)$, is defined as (2.7) and $\theta_4(t) = \left(\frac{h(y)}{P(t)}\right)'$. ω and α are positive constants which will be specified later in the proof. We have

$$V_2 = \frac{h(y)}{P(t)}F(x) + \frac{1}{2}((z + h(y)y) + \mu_2 P(t)x)^2 + \frac{\varphi(x)}{2}\left(y + \frac{f(x)}{\varphi(x)}\right)^2 - \frac{f^2(x)}{2\varphi(x)} + \frac{1}{2}\mu_2(1 - P(t)\mu_2)x^2.$$

We can verify that

$$\begin{aligned} V_2 &\geqslant \frac{h(y)}{P(t)}F(x) - \frac{f^2(x)}{2\varphi(x)} \\ &\geqslant \frac{h_0}{g_1}F(x) - \frac{1}{2\varphi_0}f^2(x) \\ &\geqslant \int_0^x \left(\frac{h_0}{g_1} - \frac{\delta_1}{\varphi_0}\right)f(u)du \\ &\geqslant \delta_4 F(x), \end{aligned}$$

where $\delta_4 = h_0 - \frac{g_1\delta_1}{\varphi_0} > 0$. Thus from (ii) we obtain,

$$V_2 \geqslant \frac{\delta_4\delta_0}{2}x^2. \tag{3.5}$$

Clearly, from (3.5) and (2.10) and the fact that the integral $\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds$ is positive, we deduce that

$$V \geqslant \delta_2 y^2 + \delta_2 z^2 + \frac{\delta_5\delta_0}{2}x^2,$$

where $\delta_5 = \delta_3 + \delta_4$. Further simplification of the last estimate gives

$$V \geqslant k(x^2 + y^2 + z^2), \tag{3.6}$$

where $k = min\{\delta_2; \frac{\delta_5\delta_0}{2}\}$. From hypotheses (i) and (iv), we have

$$\begin{aligned} \int_0^t |\theta_4(s)|ds &\leqslant \frac{1}{g_0}\int_{\psi_1(t)}^{\psi_2(t)} |h'(u)|du + \frac{h_1}{g_0^2}\int_{\sigma_1(t)}^{\sigma_2(t)} |g'(u)|du \\ &\leqslant \frac{1}{g_0}\int_{-\infty}^{+\infty} |h'(u)|du + \frac{h_1}{g_0^2}\int_{-\infty}^{+\infty} (|g'(u)| + |h'(u)|)du \\ &\leqslant M < \infty, \end{aligned}$$

where σ_1, σ_2, ψ_1, ψ_2 are defined as (2.3).

Therefore we can find a continuous function $U_1(|\Phi(0)|)$ with

$$U_1(|\Phi(0)|) \geqslant 0 \quad \text{and} \quad U_1(|\Phi(0)|) \leqslant U(t, \Phi).$$

The existence of a continuous function $U_2(\|\phi\|)$ which satisfies the inequality $U(t,\phi) \leqslant U_2(\|\phi\|)$, is also easily shown.

Using a basic calculation, the time derivative of the functional $V(t, x_t, y_t, z_t)$, along the trajectories of the system (3.3), results in

$$V'_{(3.3)} = V'_{1_{(3.3)}} + V'_{2_{(3.3)}},$$

where

$$V'_{1_{(3.3)}} = V'_{1_{(2.4)}} + \mu_1 y e(t) + \frac{z}{P(t)}e(t).$$

Combining (2.13) with (j) we get

$$V'_{1_{(3.3)}} \leqslant -\left[\mu_1\varphi_0 - \delta_1 - \left(\lambda + \frac{\mu_1\delta_1}{2}\right)r\right]y^2 - \left[\frac{h_0 - \mu_1 g_1}{g_1^2} - \frac{\delta_1 r}{2g_0}\right]z^2 + \left[\frac{\delta_1}{2}\left(\mu_1 + \frac{1}{g_0}\right) - \lambda\right]\int_{t-r}^{t} y^2(\xi)d\xi + \frac{\mu_1}{g_0}|y|m + |z|m + k_1 Q(t)(y^2 + z^2). \quad (3.7)$$

We have

$$V'_{2_{(3.3)}} = \theta_4(t)F(x) + \mu_2(1-\varphi(x))xy + \frac{f'(x)P(t) - h(y)\varphi(x)}{P(t)}y^2 + \mu_2 yz + \mu_2 h(y)y^2 - \mu_2 xf(x) + \mu_2 xe(t) + ze(t) + \frac{h(y)}{P(t)}ye(t) - \frac{1}{2}(z + h(y)y)^2\theta_1(t) - \left[\frac{1}{2}y^2 + \frac{1}{P(t)}(xz + h(y)xy) + \mu_2 x^2\right]\theta_3(t) + \left(\mu_2 x + z + \frac{h(y)}{P(t)}y\right)\int_{t-r}^{t} y(s)f'(x(s))ds.$$

We can now proceed analogously to (2.12)

$$\left(\mu_2 x + \frac{h(y)}{P(t)}y + \frac{z}{P(t)}\right)\int_{t-r}^{t} y(s)f'(x(s))ds \leqslant \left(\frac{\mu_2\delta_1 r}{2}x^2 + \frac{\delta_1 h_1 r}{2g_0}y^2 + \frac{\delta_1 r}{2g_0}z^2\right) + \left(\frac{\mu_2\delta_1}{2} + \frac{h_1\delta_1}{2g_0} + \frac{\delta_1}{2g_0}\right)\int_{t-r}^{t} y^2(\xi)d\xi.$$

Using Schwartz inequality and conditions (i), (ii), (j) we obtain

$$V'_{2_{(3.3)}} \leqslant \theta_4(t)F(x) - \mu_2\left[\delta_0 - \frac{(1+\varphi_1)}{2} - \frac{\delta_1 r}{2}\right]x^2 - \left[\frac{h_0\varphi_0 - \delta_1 g_1}{g_0} - \frac{\mu_2}{2}(2+\varphi_1)\right]y^2 + \left[\mu_2 h_1 + \frac{r\delta_1 h_1}{2g_0}\right]y^2 + \left[\frac{\mu_2}{2} + \frac{\delta_1 r}{2g_0}\right]z^2 + \frac{h_1}{g_0}|y|m + \mu_2|x|m + |z|m + (z^2 + h_1^2 y^2)|\theta_1(t)| + \left[\left(\mu_2 + \frac{1}{2g_0}(1+h_1)\right)x^2 + \frac{1}{2}y^2 + \frac{1}{2g_0}z^2\right]|\theta_3(t)| + \delta_1\frac{h_1 + \mu_2 g_0 + 1}{2g_0}\int_{t-r}^{t} y^2(\xi)d\xi. \quad (3.8)$$

Combining (3.8) and (3.7), and using condition (jjj), we get

$$V'_{(3.3)} \leqslant \theta_4(t)F(x) + \mu_2|x|m + \frac{h_1+\mu_1}{g_0}|y|m + 2|z|m - \beta_1 x^2 - \beta_2 y^2 - \beta_3 z^2 + k_2 Q(t)\left(x^2+y^2+z^2\right) + \left(\frac{\delta_1[g_0(\mu_1+\mu_2)+2+h_1]}{2g_0} - \lambda\right)\int_{t-r}^{t} y^2(\xi)d\xi,$$

where

$$\beta_1 = \mu_2\left[\delta_0 - \frac{(1+\varphi_1)}{2} - \frac{\delta_1 r}{2}\right],$$

$$\beta_2 = \mu_1\varphi_0 - \delta_1 - \mu_2 h_1 - r\left(\lambda + \frac{\delta_1\mu_1}{2} + \frac{\delta_1 h_1}{2g_0}\right),$$

$$\beta_3 = \left[\frac{h_0 - \mu_1 g_1}{g_1^2} - \frac{\mu_2}{2} - \frac{\delta_1 r}{g_0}\right],$$

$$k_2 = k_1 + \frac{1}{2g_0}(1+h_1) + \mu_2 + (1+h_1^2).$$

Taking

$$\frac{\delta_1[g_0(\mu_1+\mu_2)+2+h_1]}{2g_0} = \lambda,$$

$$\min\{\beta_1,\ \beta_2,\ \beta_3\} = \beta,$$

and using (3.2) and (3.6) and (jj) the last estimate becomes

$$V'_{(3.3)} \leqslant \theta_4(t)F(x) + \mu_2|x|m + \frac{h_1+\mu_1}{g_0}|y|m + 2|z|m - \beta\left(x^2+y^2+z^2\right) + \frac{k_2}{k}Q(t)V.$$

It follows that

$$V'_{(3.3)} \leqslant \theta_4(t)F(x) - \beta(x^2+y^2+z^2) + \beta\gamma(|x|+|z|+|y|) + \frac{k_2}{k}Q(t)V,$$

where

$$\gamma = \frac{m}{\beta}max\left\{2,\ \frac{h_1+\mu_1}{g_0},\ \mu_2\right\}.$$

The above estimate may be written as

$$\begin{aligned} V'_{(3.3)} &\leqslant \theta_4(t)F(x) - \frac{\beta}{2}(x^2+y^2+z^2) + \frac{k_2}{k}Q(t)V - \frac{\beta}{2}[x^2+y^2+z^2 - 2\gamma(|x|+|z|+|y|)] \\ &= \theta_4(t)F(x) - \frac{\beta}{2}(x^2+y^2+z^2) + \frac{k_2}{k}Q(t)V - \frac{\beta}{2}[(|x|-\gamma)^2 + (|y|-\gamma)^2 + (|z|-\gamma)^2] + \frac{3\beta}{2}\gamma^2 \\ &\leqslant \theta_4(t)F(x) - \frac{\beta}{2}(x^2+y^2+z^2) + \frac{k_2}{k}Q(t)V + \frac{3\beta}{2}\gamma^2. \end{aligned}$$

It is clear that

$$U'_{(3.3)}(t, x_t, y_t, z_t) = e^{-\eta(t)}\left[V'_{(3.3)} - \left(\frac{1}{\omega}Q(t) + \frac{1}{\alpha}|\theta_4(t)|\right)V\right].$$

Putting $\omega = \frac{k}{k_2}$ and $\alpha = \delta_4$ we obtain

$$U'_{(3.3)}(t, x_t, y_t, z_t) \leqslant L\left[-\frac{\beta}{2}(x^2+y^2+z^2) + \frac{3\beta}{2}\gamma^2\right], \quad \text{for some } L > 0.$$

Hence the conclusions of Theorem 3.2 follow from Lemma 3.1, this completes the proof of Theorem. □

4. Conclusions

It is well known that the problem of ultimate boundedness of solutions of nonlinear is very important in the theory and applications of differential equations. Sufficient conditions for the boundedness, ultimate boundedness, and the asymptotic stability of solutions for a certain third order nonlinear differential equation are given with the aid of an effective method namely Lyapunov second or direct method. The appropriate Lyapunov function is given explicitly to obtain the results. Finally, it is worth noting that our study complement some well known results on the third order differential equations in the literature.

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