Journal of the Association of Arab Universities for Basic and Applied Sciences (2016) **21**, 86–91

University of Bahrain
Journal of the Association of Arab Universities for Basic and Applied Sciences
www.elsevier.com/locate/jaaubas
www.sciencedirect.com

ORIGINAL ARTICLE

An efficient numerical method for computation of the number of complex zeros of real polynomials inside the open unit disk

Muhammad Mujtaba Shaikh [a,*], **Karem Boubaker** [b]

[a] *Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro, Sindh, Pakistan*
[b] *Tunis-City University Ecole Supérieure des Sciences et Techniques de Tunis, 63 Rue sidi Jabeur, 5100 Mahdia, Tunisia*

Received 9 February 2014; revised 18 April 2015; accepted 25 April 2015
Available online 21 May 2015

KEYWORDS

Real polynomials;
Complex zeros;
Sturm sequences;
Boubaker polynomials

Abstract In this paper, a simple and efficient numerical method is proposed for computing the number of complex zeros of a real polynomial lying inside the unit disk. The proposed protocol uses the Boubaker polynomial expansion scheme (BPES) to build sequence of polynomials based on the concept of Sturm sequences. The method is used in a direct way without using any restrictions in reference to other existing methods. The protocol is applied to some example polynomials of different orders and utility of the algorithm is noticed.

1. Introduction

Computation of the number of non-real zeros of real polynomials inside the open unit disk is very important in complex analysis and system control. For example, for a corrector of the form:

$$u_{n+1} = \sum_{i=0}^{k-1} A_i u_{n-i} + h \sum_{i=-1}^{k-1} a_i u'_{n-i} \tag{1}$$

Simpson's stability rule is ensured if the polynomial:

$$P(z) = z^k - \sum_{i=0}^{k-1} A_i z^{k-1-i} \tag{2}$$

has all of its zeros in the open unit disk.

The Newton mapping of non-zero polynomials is also based on this notion. In fact, for a given polynomial $P(z)$, the Newton mapping $N_P(z)$, which is defined by:

$$N_P(z) = z - P(z)/P'(z) \tag{3}$$

would be defined only if the zeros of $P(z)$ are contained in the open unit disk.

In this study, we present a new protocol for determining the exact number of complex zeros of a given real polynomial in the unit disk using a well-known applied mathematics protocol, the Boubaker polynomials. The polynomials were established by Boubaker (2007, 2008) and have been worked upon by many researchers till now for further developments and its utilities are being investigated to deal various types of case-studies in applied engineering, medical sciences, etc.

Several properties and modified versions of these polynomials have been investigated; to mention a few studies: Boubaker et al. (2007), Labiadh (2007), Oyodum et al. (2009), Zhao et al. (2009, 2010) and Barry and Hennessy (2010). A modified

* Corresponding author. Cell: +92 333 2617602.
E-mail address: mujtaba.shaikh@faculty.muet.edu.pk (M.M. Shaikh).
Peer review under responsibility of University of Bahrain.

http://dx.doi.org/10.1016/j.jaubas.2015.04.005
1815-3852 © 2015 University of Bahrain. Publishing services by Elsevier B.V.

version of these polynomials, called $4q$-Boubaker polynomials, was the basis for the development of the Boubaker polynomials expansion scheme (BPES). The scheme has been used by Agida and Kumar (2010) and Kumar (2010) to solve particular integral equations. On the other hand few standard boundary value problems of ordinary differential equations (Boubaker, 2008; Zhang and Naing, 2010; Koçak et al., 2011a) and many physical models involving ordinary differential equations systems (Milgram, 2011; Dubey et al., 2010, Yildirim et al., 2010) were solved more efficiently by BPES as compared to other methods. Physical models in terms of partial differential equations in many fields were reliably addressed through BPES. For example: the works carried out by Ghrib et al. (2008), Guezmir et al. (2009) and Zhang and Li (2010) in general to investigate material and alloy properties and more particularly the works by Zhang (2010b) and Slama et al. (2008a,b, 2009, 2010) in the field of resistance spot welding research to obtain analytical temperature distribution.

The contributions by Ghanouchi and Labiadh (2008), Tabatabaei et al. (2009), Belhadj et al. (2009) and Koçak et al. (2011b) further evoked the use of BPES to solve core studies in the field of Heat and Mass Transfer. Awojoyogbe and Boubaker (2009) and in many other studies jointly explained how NMR blood flow equations can be solved in various heart models to find magnetic phase shift, and in Bio-medical engineering to find net magnetization under the MRI exposure in various geometries. The work carried out by Fridjine and Amlouk (2009) discusses the case of optimizing functional materials in hybrid solar energy devices.

The main idea in this paper consists of constructing the Sturm-sequences which are built using the properties of BPES. The idea of this construction is based on the work of Schelin (1983) who first used Chebyshev polynomials to construct Sturm-like sequence to count zeros of real polynomials. A similar construction using Chebyshev polynomials appears in the works of Locher and Skrzipek (1995) and Gleyse (1997). The examination of the number of sign changes and the sign repetitions in the built-off Sturm sequences in this work using $4q$-Boubaker polynomials finally leads to define the complete protocol to achieve the goal of computing the number of complex zeros of real polynomials.

The concept of sign changes and sign repetitions dates back to Seventeenth century when Rene Descarte proposed a rule of signs to find upper bound on the count of positive and negative real zeros of a polynomial. Another concept of examining signs appears in Routh–Hurwitz test and its extensions (Gantmacher, 1960) which is used to determine if all zeros of a real polynomial lie in the open left-half plane and hence to comment on polynomial stability. However, the criterion of counting the number of sign changes and the sign repetitions used to develop method in this paper is based on a similar concept used in Sturm theorem (Collins and Rudiger, 1983) to count real zeros of polynomials defined in interval $[-1, 1]$. We demonstrate through worked out examples in Section 3 that the proposed protocol – which uses little extension of the concepts in the Sturm theorem – yields encouraging results when it comes to count the number of complex zeros of real polynomials in the open unit disk.

The structure of this paper is as follows:

We begin by introducing, in Section 2, some necessary definitions and mathematical preliminaries of the Boubaker Polynomials which are required for establishing our results. This follows the procedure of constructing the Sturm-like shaped sequence of the polynomials. In Section 3, we use the protocol to determine the exact number of complex zeros of some variable degree polynomials in the open unit disk. We end with illustrating conclusion and future work.

2. Materials and methods

2.1. The Boubaker polynomials

The first monomial definition of the Boubaker polynomials (Boubaker, 2007, 2008; Ghanouchi and Labiadh, 2008; Belhadj et al., 2009) appeared in a physical study that yielded an analytical solution to the heat equation inside a physical model.

Definition 1. Boubaker polynomials monomial definition is given by:

$$B_n(X) = \sum_{p=0}^{\xi(n)} \left[\frac{(n-4p)}{(n-p)} C_{n-p}^{p} \right] \cdot (-1)^p \cdot X^{n-2p} \tag{4}$$

where:

$$\xi(n) = \left\lfloor \frac{n}{2} \right\rfloor = \frac{2n + ((-1)^n - 1)}{4}$$

(The symbol: $\lfloor\,\rfloor$ designates the floor function).

The Boubaker polynomials have also the explicit monic expression:

$$B_n(X) = X^n - (n-4).X^{n-2} + \sum_{p=2}^{\xi(n)} \left[\frac{(n-4p)}{p!} \prod_{j=p+1}^{2p-1} (n-j) \right] \cdot (-1)^p \cdot X^{n-2p} \tag{5}$$

Theorem 1. The characteristic recurrence relation for the Boubaker polynomials is:

$$B_m(X) = X \cdot B_{m-1}(X) - B_{m-2}(X) \quad \text{for :} \quad m > 2$$

Proof. For $m > 2$: $B_{m-1}(X) = \sum_{p=0}^{\xi(m-1)} \left[\frac{(m-1-4p)}{(m-1-p)} C_{m-1-p}^{p} \right] \cdot (-1)^p \cdot X^{m-1-2p}$, and: $B_{m-2}(X) = \sum_{p=0}^{\xi(m-2)} \left[\frac{(m-2-4p)}{(m-2-p)} C_{m-2-p}^{p} \right] \cdot (-1)^p \cdot X^{m-2-2p}$. By calculating the amount: $\Delta = X \cdot B_{m-1}(X) - B_{m-2}(X)$, it gives:

$$\Delta = X^m \left[\sum_{p=0}^{\xi(m-1)} \left[\frac{(m-1-4p)}{(m-1-p)} C_{m-1-p}^{p} \right] \cdot (-1)^p \cdot X^{-2p} - \sum_{p=0}^{\xi(m-2)} \left[\frac{(m-2-4p)}{(m-2-p)} C_{m-2-p}^{p} \right] \cdot (-1)^p \cdot X^{-2-2p} \right]$$

which gives:

$$\Delta = X \cdot B_{m-1}(X) - B_{m-2}(X) = X^m \sum_{p=0}^{\xi(n)} \left[\frac{(n-4p)}{(n-p)} C_{n-p}^{p} \right] \cdot (-1)^p \cdot X^{-2p} = B_m(X) \qquad \square$$

The ordinary generating function of the Boubaker polynomials is:

$$f_{\tilde{B}}(X,t)=\frac{1+3t^2}{1+t(t-X)} \tag{6}$$

Zhao et al. (2010) investigated some special properties of the Boubaker polynomials B_n for the case $n=4q$ which include involvement of only even powers of x in the polynomials and removal of the $2q$ rank monomial terms from the explicit form. In particular, these properties lead to explicit expressions with only $2q$ effective terms and hence to a class of polynomials which are all even functions. Correspondent $4q$-order Boubaker polynomials (Zhao et al., 2010) are presented in Eq. (7) as a general form and Eq. (8) as first functions:

$$B_{4q}(X)=4\sum_{p=0}^{2q}\left[\frac{(q-p)}{(4q-p)}C_{4q-p}^{p}\right]\cdot(-1)^p\cdot X^{2(2q-p)} \tag{7}$$

$$\begin{cases} B_0(X)=1; \\ B_4(X)=X^4-2; \\ B_8(X)=X^8-4X^6+8X^2-2; \\ B_{12}(X)=X^{12}-8X^{10}+18X^8-35X^4+24X^2-2; \\ B_{16}(X)=X^{16}-12X^{14}+52X^{12}-88X^{10}+168X^6-168X^4+48X^2-2; \\ B_{20}(X)=X^{20}-16X^{18}+102X^{16}-320X^{14}+455X^{12}-858X^8+1056X^6 \\ -495X^4+80X^2-2; \\ \ldots \end{cases} \tag{8}$$

The proposed protocol in this paper is based on $4q$-order Boubaker polynomials instead of original polynomials B_n due to the benefits that all $4q$-order polynomial are even functions and result in less computational cost (to be elaborated in Section 3). We quote the following important results of $4q$-Boubaker polynomials (Zhao et al., 2010) which will be useful in the construction of the Sturm shaped sequences and the final implementation of the protocol to follow. Readers can refer to (Zhao et al., 2010) for detailed proofs.

Theorem 2. The following equality holds:

$$\sum_{k=0}^{n}B_k(x)B_k(y) = 3+\frac{B_{n+1}(x)B_n(y)-B_n(x)B_{n+1}(y)}{x-y} \quad \text{for all} \quad x\neq y$$

Proof. As a consequence of recurrence relation (Theorem 1) and assuming:

$$B_k(x)B_k(y)=\frac{\Delta_k-\Delta_{k-1}}{x-y} \quad \text{for} \quad k=2,3,\ldots \tag{8}$$

where: $\Delta_k=B_{k+1}(x)B_k(y)-B_k(x)B_{k+1}(y)$ (8) summed from 0 to n gives the desired formula. □

If $x\to y$ in (8), we obtain the following Corollary.

Corollary 1. The following equality is satisfied

$$\sum_{k=0}^{n}B_k^2(x)=3+B'_{n+1}(x)B_n(x)-B'_n(x)B_{n+1}(x), n>0 \tag{9}$$

2.2. Built-off Sturm shaped sequence

Definition 2. A Sturm shaped sequence of polynomials is a set:

$$\{P_0(x),P_1(x),P_2(x),\ldots,P_M(x)\} \tag{10}$$

with P_0,P_1 and P_2 three initializing nonzero polynomials, M a given integer and $P_i|_{i=1\ldots M}$ verifying:

$$P_i(x)=\Phi_i(x)P_{i-1}(x)+P_{i-2}(x), \quad i\geqslant 2 \tag{11}$$

where $\Phi_i(x)|_{i=2\ldots N}$ is a given polynomial sequence.

Let us consider a real polynomial $Q(x)=\sum_{i=0}^{N}\xi_i x^i$, along with the sequence $\{P_0(x),P_1(x),P_2(x),\ldots,P_N(x)\}$:

$$\begin{cases} P_0(x)=\sum_{i=0}^{n}\xi_i B_{4i}(x) \\ P_1(x)=\sum_{i=1}^{n}\xi_i B_{4i-4}(x) \\ P_2(x)=B_4(x)P_1(x)-P_0(x) \\ \ldots \\ P_{k+1}(x)=B_4(x)P_k(x)-P_{k-1}(x) \\ P_N(x)=B_4(x)P_{N-1}(x)-P_{N-2}(x) \end{cases} \tag{12}$$

Here N is order of the real polynomial $Q(x)$ and n is number of non-zero terms in $Q(x)$.

Theorem 3. The sequence $\tilde{P}_N(x)=\{P_0(x),P_1(x),P_2(x),\ldots,P_N(x)\}$ with polynomials as in (12) is a Sturm shaped sequence constructed from Boubaker polynomials.

Proof. We have, for all values of $0\leqslant k<M$:

$$P_{k-1}(x)=B_4(x)P_k(x)-P_{k+1}(x)=\Lambda P_k(x)+r,$$

the remainder r of the Euclidian division of $P_{k-1}(x)$ by $P_k(x)$ is hence: $r=-P_{k+1}(x)$. □

Proposed protocol. For a sequence $\tilde{P}_N(x)=\{P_0(x),P_1(x),P_2(x),\ldots,P_N(x)\}$, associated to a polynomial $Q(x)=\sum_{i=0}^{N}\xi_i x^i$, defined in the domain $[-1,1]$, the number $Z_{1,Q}$ of complex zeros inside the unit disk is given by:

$$Z_{1,Q}=S^*(-1)+S^*(1) \tag{13}$$

where $S^*(x)=S^C(x)-S^R(x)$ represents the difference between the number of sign changes and sign repetitions in the sequence $\tilde{P}_N(x)$.

This protocol is an extension of the Sturm theorem for real zeros of real-coefficient polynomials. For a proof, refer to Collins and Rudiger (1983). While the usual Sturm theorem and related works on Sturm-like sequence using Chebyshev polynomials in literature (Schelin, 1983; Locher and Skrzipek, 1995; Gleyse, 1997) target only the number of real zeros of real coefficient polynomials in open unit disk or other annulus, we demonstrate through examples in the next section that the proposed protocol – an extension to the theorem – can be used to count number of complex zeros of real polynomials in open unit disk.

Since the built-off Sturm sequence $\tilde{P}_N(x)$ is constructed through $4q$-Boubaker polynomials, which are all even, as a consequence the number of sign changes and also the sign repetitions at -1 and 1 will be the same, i.e. $S^C(-1) = S^C(1)$ and $S^R(-1) = S^R(1) \Rightarrow S^*(-1) = S^*(1)$. Thus, (13) can equivalently be expressed as:

$$Z_{1,Q} = 2S^*(-1) = 2S^*(1) \tag{14}$$

It can be noted that the use of $4q$-Boubaker polynomials minimizes the computational cost of (13) by half as one needs to count the sign changes and repetitions either only at 1 or -1 as in (14).

3. Results and discussion

The described protocol has been applied on following polynomials (all zeros are shown opposite):

Example 1:

$$Q_1(x) = x^8 + \frac{9}{8}x^6 + \frac{281}{256}x^4 - \frac{225}{512}x^2 + \frac{625}{4096}\left(\frac{1}{2} \pm i, -\frac{1}{2} \pm i, \frac{1}{2} \pm \frac{1}{4}i, -\frac{1}{2} \pm \frac{1}{4}i\right)$$

Example 2:

$$Q_2(x) = x^5 + 4x^4 + \frac{15}{2}x^3 + \frac{35}{2}x^2 + 14x + 6\left(-3, \pm 2i, -\frac{1}{2} \pm \frac{1}{2}i\right)$$

Example 3:

$$Q_3(x) = x^7 + \frac{1}{2}x^5 - x^3 - \frac{1}{2}x\left(0, \pm 1, \pm i, \pm \frac{1}{\sqrt{2}}i\right)$$

Example 4:

$$Q_4(x) = x^3 - x^2 + \frac{1}{4}x - \frac{1}{4}\left(1, \pm \frac{1}{2}i\right)$$

Implementation details of the proposed protocol on polynomials in Examples 1–4 are given in Table 1 with specific values, sign sequence and sign patterns in corresponding Sturm-shaped sequences. We explicitly describe implementation on, say, $Q_1(x)$. The application of the protocol on $Q_1(x)$ gives the following Boubaker polynomial built Sturm-shaped sequence:

$$\begin{cases} P_0(x) = \sum_{i=0}^{n} \xi_i B_{4i}(x) = B_{16}(x) + \frac{9}{8}B_{12}(x) + \frac{281}{256}B_8(x) \\ -\frac{225}{512}B_4(x) + \frac{625}{4096}B_0(x) \\ P_1(x) = \sum_{i=1}^{n} \xi_i B_{4i-4}(x) = B_{12}(x) + \frac{9}{8}B_8(x) + \frac{281}{256}B_4(x) - \frac{225}{512}B_0(x) \\ P_{k+1}(x) = B_4(x)P_k(x) - P_{k-1}(x), \ \ k = 2, 3, \ldots, 8 \end{cases}$$

and corresponding sign sequence $\{+, -, -, +, -, -, +, -, -\}$ at $x = 1$ or $x = -1$. Consequently:

$$Z_{1,Q1} = 2S^*(-1) = 2S^*(1) = 2(5 - 3) = 4$$

which is true as only four complex zeros of $Q_1(x)$: $\frac{1}{2} + i, \frac{1}{2} - i, -\frac{1}{2} + i$ and $-\frac{1}{2} - i$ lie in the open unit disk. Zeros loci for $Q_1(x)$ are shown in Fig. 1.

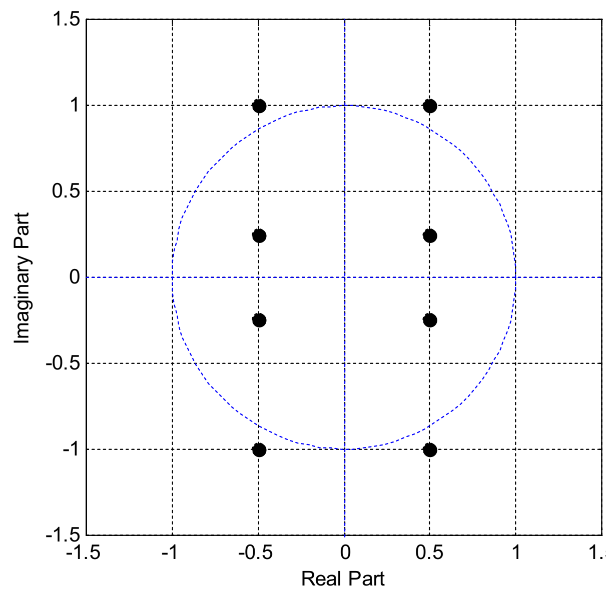

Fig. 1 Zeros loci for $Q_1(x)$.

Table 1 Implementation details of the proposed protocol on example polynomials 1–4.

	$Q_1(x)$	$Q_2(x)$	$Q_3(x)$	$Q_4(x)$
P_0	341/537	57/2	0	−11/2
P_1	−83/512	10	3/2	17/4
P_2	−1143/2414	−77/2	−3/2	5/4
P_3	341/537	57/2	0	−11/2
P_4	−83/512	10	3/2	–
P_5	−1143/2414	−77/2	−3/2	–
P_6	341/537	–	0	–
P_7	−83/512	–	3/2	–
P_8	−1143/2414	–	–	–
Sign sequence at $x = 1$ or -1	$\{+, -, -, +, -, -, +, -, -\}$	$\{+, +, -, +, +, -\}$	$\{+, +, -, +, +, -, +, +\}$	$\{-, +, +, -\}$
$S^C(-1)$ or $S^C(1)$	5	3	4	2
$S^R(-1)$ or $S^R(1)$	3	2	3	1
$S^*(-1)$ or $S^*(1)$	2	1	1	1
$Z_{1,Q}$	**4**	**2**	**2**	**2**

Bold values represent main output of the proposed protocol on examples 1–4. These bold values represent the number of complex zeros inside the unit disk, found by the protocol for polynomials in examples 1–4.

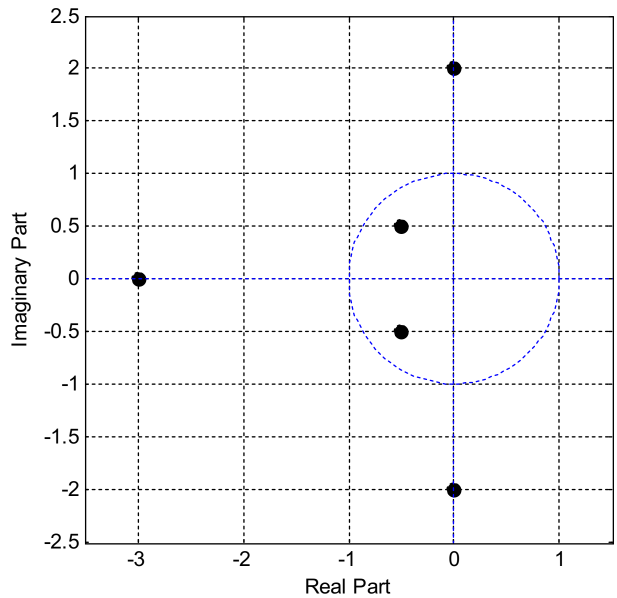

Fig. 2 Zeros loci for $Q_2(x)$.

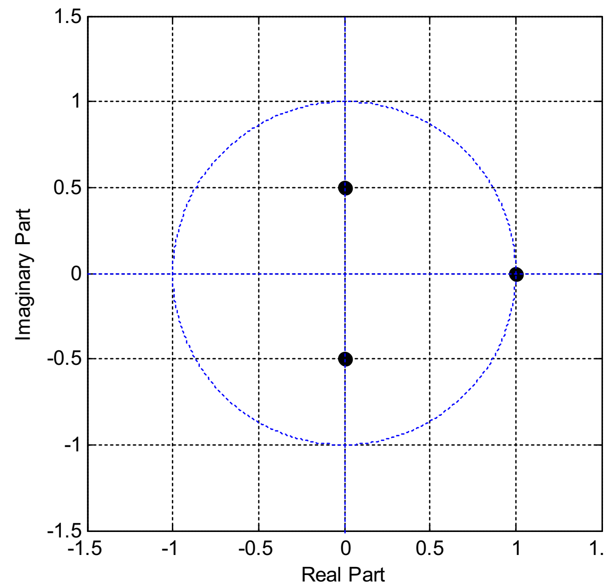

Fig. 4 Zeros loci for $Q_4(x)$.

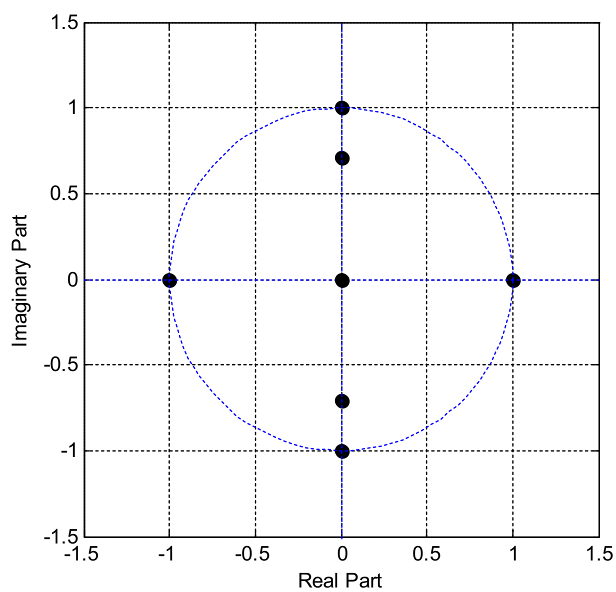

Fig. 3 Zeros loci for $Q_3(x)$.

The results in Table 1 speak for themselves. The proposed protocol shows that polynomials in Examples 1–4 have 4, 2, 2, and 2 complex zeros inside the open unit disk, respectively, which is in good agreement with the loci plots of these polynomials in z-plane (Figs. 1–4). It can be noted that the proposed method counts only those complex zeros inside open unit disk that are purely non-real, i.e. involve some imaginary term. This is evident through Example-3 and meanwhile from Fig. 3 that $x = 0$ (a purely real zero of $Q_3(x)$) – beside located in the open unit disk – is not counted by the proposed algorithm.

4. Conclusion and future work

The exact number of complex zeros in the open unit disk of some polynomials of different orders has been determined using the Boubaker polynomial generated Sturm sequence. The protocol is general and efficient since no restriction is applied to the targeted polynomial. According to the example investigations, this method is a simple and efficient numerical method for computing the number of polynomials complex zeros lying inside the unit disk.

The construction of Boubaker polynomials built-off Sturm shaped sequences in this work to exactly compute the number of complex zeros for real polynomials, as a start, through this work further encourages researchers to revisit the approximations, work to refine it for other similar problems and devise extended methods. Investigation of further accuracy and suitability of this protocol along with proposition of its utility to address case studies in complex analysis and control theory are the topics of future research. It can be observed that no conditions are presumed for the polynomial $Q(x)$, as opposed to the methods (Gleyse, 1997; Gleyse and Larabi, 2011). Comparison of our method will be made with those of the Cauchy-indices-related method, used by Gleyse (1997), or those of methods using Schur-Cohn, Brown and Cohn transforms (Gleyse and Larabi, 2011) from the view-point of analysis of order of complexity in future.

Acknowledgement

Authors are thankful to their institutes for providing facilities for conducting this research. Authors are also highly indebted to referees for highlighting important gaps and key points to adequately improve accuracy in the present version of our work.

References

Agida, M., Kumar, A.S., 2010. A Boubaker polynomials expansion scheme solution to random Love's equation in the case of a rational Kernel. Electron. J. Theor. Phys. 7 (24), 319–326.

Awojoyogbe, O.B., Boubaker, K., 2009. A solution to Bloch NMR flow equations for the analysis of hemodynamic functions of blood flow system using m-Boubaker polynomials. Current Appl. Phys. 9 (1), 278–283.

Barry, P., Hennessy, A., 2010. Meixner-type results for Riordan arrays and associated integer sequences. J. Integer Seq. 13, 1–34.

Belhadj, A., Onyango, O.F., Rozibaeva, N., 2009. Boubaker polynomials expansion scheme-related heat transfer investigation inside keyhole model. J. Thermophys. Heat Transfer 23 (3), 639–640.

Boubaker, K., 2007. On modified Boubaker polynomials: some differential and analytical properties of the new polynomials issued from an attempt for solving bi-varied heat equation. Trends Appl. Sci. Res. 2 (6), 540–544.

Boubaker, K., 2008. The Boubaker polynomials, a new function class for solving bi-varied second order differential equations. FEJ Appl. Math. 31 (3), 273–436.

Boubaker, K., Chaouachi, A., Amlouk, M., Bouzouita, H., 2007. Enhancement of pyrolysis spray disposal performance using thermal time-response to precursor uniform deposition. Eur. Phys. J. Appl. Phys. 37 (01), 105–109.

Collins, George E., Rudiger, Loos, 1983. Real zeros of polynomials. In: Computer Algebra. Springer Vienna, pp. 83–94.

Dubey, B., Zhao, T.G., Jonsson, M., Rahmanov, H., 2010. A solution to the accelerated-predator-satiety Lotka–Volterra predator–prey problem using Boubaker polynomial expansion scheme. J. Theor. Biol. 264 (1), 154–160.

Fridjine, S., Amlouk, M., 2009. A new parameter: an abacus for optimizing pv–t hybrid solar device functional materials using the Boubaker polynomials expansion scheme. Mod. Phys. Lett. B 23 (17), 2179–2191.

Gantmacher, Felix R., 1960. In: Theory of Matrices, vol. 2. Chelsea Publishing Company.

Ghanouchi, J., Labiadh, H., 2008. An Attempt to solve the heat transfer equation in a model of pyrolysis spray using 4q-order Boubaker polynomials. Int. J. Heat Technol. 26 (1), 49–53.

Ghrib, T., Boubaker, K., Bouhafs, M., 2008. Investigation of thermal diffusivity–microhardness correlation extended to surface-nitrured steel using Boubaker polynomials expansion. Mod. Phys. Lett. B 22 (29), 2893–2907.

Gleyse, B., 1997. Sturm sequences and the number of zeros of a real polynomial in the unit disk: numerical computation. Appl. Math. Lett. 10 (2), 123–127.

Gleyse, B., Larabi, A., 2011. Numerical computation of the number of zeros of real polynomials in the open unit disk using a Chebyshev polynomials representation. Appl. Math. Lett. 24 (5), 598–600.

Guezmir, N., Nasrallah, T.B., Boubaker, K., Amlouk, M., Belgacem, S., 2009. Optical modeling of compound CuInS< sub> 2</sub> using relative dielectric function approach and Boubaker polynomials expansion scheme BPES. J. Alloys Compd. 481 (1), 543–548.

Koçak, H., Yıldırım, A., Zhang, D.H., Mohyud-Din, S.T., 2011a. The comparative Boubaker polynomials expansion scheme (BPES) and homotopy perturbation method (HPM) for solving a standard nonlinear second-order boundary value problem. Math. Comput. Model. 54 (1), 417–422.

Koçak, H., Dahong, Z., Yildirim, A., 2011b. A range-free method to determine Antoine vapor-pressure heat transfer-related equation coefficients using the Boubaker polynomial expansion scheme. Russ. J. Phys. Chem. A 85 (5), 900–902.

Kumar, A.S., 2010. An analytical solution to applied mathematics-related Love's equation using the Boubaker polynomials expansion scheme. J. Franklin Inst. 347 (9), 1755–1761.

Labiadh, H., 2007. A Sturm–Liouville shaped characteristic differential equation as a guide to establish a quasi-polynomial expression to the Boubaker polynomials. Differ. Uravn. Protsessy Upr., 117–133.

Locher, F., Skrzipek, M.R., 1995. An algorithm for locating all zeros of a real polynomial. Computing 54 (4), 359–375.

Milgram, A., 2011. The stability of the Boubaker polynomials expansion scheme (BPES)-based solution to Lotka–Volterra problem. J. Theor. Biol. 271 (1), 157–158.

Oyodum, O.D., Awojoyogbe, O.B., Dada, M., Magnuson, J., 2009. On the earliest definition of the Boubaker polynomials. Eur. Phys. J. Appl. Phys. 46, 21201–21203.

Schelin, C.W., 1983. Counting zeros of real polynomials within the unit disk. SIAM J. Numer. Anal. 20 (5), 1023–1031.

Slama, S., Bouhafs, M., Mahmoud, K.B., 2008a. A Boubaker polynomials solution to heat equation for monitoring A3 point evolution during resistance spot welding. Int. J. Heat Technol. 26 (2), 141–146.

Slama, S., Bessrour, J., Boubaker, K., Bouhafs, M., 2008b. A dynamical model for investigation of A3 point maximal spatial evolution during resistance spot welding using Boubaker polynomials. Eur. Phys. J. Appl. Phys. 44 (03), 317–322.

Slama, S., Bessrour, J., Bouhafs, M., Mahmoud, K.B., 2009. Numerical distribution of temperature as a guide to investigation of melting point maximal front spatial evolution during resistance spot welding using Boubaker polynomials. Numer. Heat Transfer A Appl. 55 (4), 401–408.

Slama, Boubaker, J., Bessrour, K., Bouhafs, M., 2010. A dynamical model for investigation of A3 point maximal spatial evolution during resistance spot welding using Boubaker polynomials; on the higher order derivatives of the 4q-Boubaker polynomials. Eur. Phys. J. Appl. Phys. 50 (01), 11201–11203.

Tabatabaei, S.A.H.A., Zhao, T., Awojoyogbe, O.B., Moses, F.O., 2009. Cut-off cooling velocity profiling inside a keyhole model using the Boubaker polynomials expansion scheme. Heat Mass Transfer 45 (10), 1247–1251.

Yildirim, A., Mohyud-Din, S.T., Zhang, D.H., 2010. Analytical solutions to the pulsed Klein–Gordon equation using modified variational iteration method (MVIM) and Boubaker polynomials expansion scheme (BPES). Comput. Math. Appl. 59 (8), 2473–2477.

Zhang, D.H., 2010. Comment on "A dynamical model for investigation of A3 point maximal spatial evolution during resistance spot welding using Boubaker polynomials" by S. Zhang.

Zhang, D.H., Li, F.W., 2010. Boubaker polynomials expansion scheme (BPES) optimisation of copper tin sulfide ternary materials precursor's ratio-related properties. Mater. Lett. 64 (6), 778–780.

Zhang, D.H., Naing, L., 2010. The Boubaker polynomials expansion scheme BPES for solving a standard boundary value problem. Appl. Sci. 12, 153–157.

Zhao, T., Mahmoud, B.B., Toumi, M.A., Faromika, O.P., Dada, M., Awojoyogbe, O.B., Lin, F., 2009. Some new properties of the applied-physics related Boubaker polynomials. Differ. Eqns. Control Process 1, 7–19.

Zhao, T.G., Naing, L., Yue, W.X., 2010. Some new features of the Boubaker polynomials expansion scheme BPES. Math. Notes 87 (1–2), 165–168.