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بعض معادلات الفروق التفاضلية اللاخطية في علم الفيزياء

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الملخص:

يستخدم في هذا البحث طريقة دالة (tanh) المتقطعة مع تطبيقها رمزيا باستخدام الحاسب لإيجاد حلول موجية مسافرة جديدة مضبوطة لبعض معادلات الفروق التفاضلية اللاخطية ذات الاهتمام في علم الفيزياء مثل: معادلات (Toda Lattice ،Hybrid)، ومعادلة فروق (Toda Lattice) النسبية.

النتائج تظهر وجود عدة أنواع من الحلول المضبوطة: سولوتنية، دورية، كسرية. يمكن استخدام الطريقة لمعادلة فروق تفاضلية أخرى.

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ORIGINAL ARTICLE

On the nonlinear difference-differential equations arising in physics

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Abstract Here, an extended discrete tanh function method with a computerized symbolic computation is used constructing a new exact travelling wave solutions of nonlinear differential difference equations of special interest in physics, namely, Hybrid equation, Toda lattice equation and Relativistic Toda lattice difference equations.

As a result, we obtain many kinds of exact solutions which include soliton solutions, periodic solutions and rational solutions in a uniform way if solutions of these kinds exist. The method is straightforward and concise, and it can also be applied to other nonlinear difference differential equations in physics.

1. Introduction

A large class of nonlinear evolution equations have been derived and widely applied in various branches of natural science. The investigation of the travelling wave solutions for nonlinear evolution equations arising in physics plays an important role in the study of nonlinear physical phenomena. A variety of powerful methods for obtaining the exact solutions of nonlinear evolution equations have been presented (Abdou and Soliman, 2005; Abdou, 2008, 2007, 2008a,b; Abdou and Zhang, 2009; Abulwafa et al., 2007, 2008; He and Abdou, 2007; He, 2006).

The nonlinear differential-difference equations (DDEs) have been the focus of many nonlinear studies. DDEs describe many important phenomena and dynamical processes in many different fields, such as particle vibrations in lattices, currents in electrical networks, pulses in biological chains (Ablowitz and Clarson, 1991; Kevrekidis et al., 2001; Tsuchida et al., 1999; Hirota, 2004; Qian et al., 2004; Ma and Geng, 2001) and so on.

DDEs play an important role in the study of modern physics and also play a crucial role in numerical simulations of nonlinear partial differential equations (NLPDEs), queueing problems, and discretization in solid state and quantum physics.

There have been developed many methods to solve DDEs, such as inverse scattering method (Tsuchida et al., 1999) and Hirota bilinear method (Hirota, 2004), Variables separate method (Qian et al., 2004), Buacklund transformation (Ma and Geng, 2001) and Darboux transformation can also be

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applied to solve DDEs and other methods (Xie and Wang, 2009; Dai and Zhang, 2006; Ahmet, 2009; Wang and Zhang, 2007; Baldwin et al., 2004; Wang, 2009).

The rest of the paper is organized as follows: In Section 2, extended discrete tanh-function method is presented. In Section 3, we choose three nonlinear difference differential equations, namely, Hybrid equation, Toda lattice equation and Relativistic Toda lattice difference equations to illustrate the validity and advantage of this method. Finally conclusion and discussion are given in Section 4.

2. Methodology

In what follows is the summarized discrete tanh method (Wang, 2009). For a given nonlinear difference equations as

$$\psi_{n+1} = \frac{\psi_n + \epsilon A}{1 + \delta A \psi_n}, \tag{1}$$

$$\psi_{n-1} = \frac{\psi_n - \epsilon A}{1 - \delta A \psi_n^2},$$

$$\frac{\partial \psi_n}{\partial \xi_n} = \epsilon - \delta \psi_n^2, \tag{2}$$

where $\xi_n = kn + \lambda t$. For a given $\epsilon = 1$, $\delta = 1$, $A = tanh(k)$, the equation system has solutions $\psi_n = tanh(\xi_n)$ and $\psi_n = coth(\xi_n)$. In case of the parameters taken as $\epsilon = 1, \delta = -1, A = tan(k)$, the equation system has solutions $\psi_n = tan(\xi_n)$ and $\psi_n = -cot(\xi_n)$. For the parameters taken as $\epsilon = 0$, $\delta = 1, A = k$, the equation system has solutions $\psi_n = \frac{1}{\xi_n}$.

For a given nonlinear difference differential equations as

$$\phi(u_{n+p_1}(x), u_{n+p_2}(x), u_{n+p_3}(x), \ldots, u_{n+p_k}(x), u'_{n+p_1}(x), u'_{n+p_2}(x), .., u'_{n+p_k}(x), .$$

$$, u^r_{n+p_1}(x), u^r_{n+p_2}(x), \ldots, u^r_{n+p_k}) = 0, \tag{3}$$

where the dependent variable u_n have N components $u_{i,n}$ and so do its shifts; the continuous variable x has h components x, the discrete variable n has s components n_j; the k shift vectors p_i, and $u^r(x)$ denotes the collection of mixed derivative terms of order r. By introducing wave transformation

$$u_n(x) = U(\xi_n) = U_n, \xi_n = \sum_{j=1}^{s} k_i n_i + \sum_{j=1}^{h} \lambda_j x_j + c,$$

$$u_{n+p}(x) = U(\xi_{n+p}) = U_{n+p}, \xi_{n+p} = \sum_{j=1}^{s} k_i(n_i + p) + \sum_{j=1}^{h} \lambda_j x_j + c, \tag{4}$$

where, k_i and λ_j are all arbitrary constants to be determined later, c is constant.

In the context of discrete tanh function method, many authors (Baldwin et al., 2004; Wang, 2009) used the ansatz

$$U(\xi_n) = \sum_{i=0}^{M} a_i \psi_n^i(\xi_n), \tag{5}$$

where a_i are constants to be determined. In order to construct more general, it is reasonable to introduce the following ansatz

$$U(\xi_n) = \sum_{i=0}^{M} a_i \psi_n^i(\xi_n) + \sum_{j=1}^{M} b_j \psi_n^{-j}(\xi_n) \tag{6}$$

It is observed that the solution of the ansatz (8) goes back to that obtained from (7) once $b_j = 0$, where b_j are constants to be determined later, ψ_n satisfy the system of Eqs. (1) and (2), M can be determined.

Case(1): We set the degree of $U_{n+p} - U_{n-p}, p \neq 0$ is zero. For the other terms in DDE, U_n is of degree M_i is ψ_n and U_{n+p}, $p \neq 0$ is of degree zero. Then we can balance the highest nonlinear terms and the highest linear terms to determine M in Eq. (7). If M is odd, then give the value of M to m_1

Case(2): Replacing zero by -1 as the degree of $U_{n+p} - U_{n-p}$, $p \neq 0$, we balance the term again. If M is even, then give the value of M to m_2, otherwise omit it.

The polynomial form expression of difference terms $U_{n+p} = \sum_{i=0}^{M} a_i \psi_{n+p}^i + \sum_{j=1}^{M} b_j \psi_{n+p}^{-j}$ in the similar way as Eq. (6). Inserting the expression of $U_n, \ldots\ldots, U_{n+p}$ in to Eq. (5) yields an ordinary DDE in terms of $\psi_n, \ldots, \psi_{n+p}$. With the aid of Eqs. (1)–(3), we reduce the equation obtained.

Collecting coefficients of all terms, $\psi_n^i, i = 1, 2, \ldots..$ and setting to zero yield a system of algebraic equation.

Solving the system of algebraic equations by computer algebra systems such as Maple.

Substituting the result relation obtained above and combining the solutions of Eqs. (1)–(3), we could get the solutions of given DDE Eq. (4)

3. New applications

To illustrate the effectiveness and the advantages of the proposed method, three models of nonlinear differential difference equations in physics are chosen, namely, Hybrid equation, Toda lattice equation and Relativistic Toda lattice difference equations. As a result, many exact travelling wave solutions are obtained including solitary wave solutions expressed by hyperbolic functions, periodic solutions expressed by trigonometric functions and rational solutions.

3.1. Example(1). Hybrid nonlinear difference differential equation

Let us first consider the Hybrid nonlinear differential difference equation (Baldwin et al., 2004),

$$\frac{\partial u_n}{\partial t} = \left(1 + \alpha u_n + \beta u_n^2\right)(u_{n+1} - u_{n-1}), \tag{7}$$

where α and β are constants. The Hybrid nonlinear difference Eq. (9) describes the discretization of the KdV and modified KdV equations. Making use the travelling wave solution as

$$u(n, t) = U(\xi_n), \quad \xi_n = kn + \lambda t + c, \tag{8}$$

where k, c and λ are constants. Then the discrete Hybrid nonlinear difference equation reduces to

$$\lambda \frac{\partial U_n}{\partial \xi_n} = \left(1 + \alpha U_n(\xi_n) + \beta U_n^2(\xi_n)\right)(U_{n+1}(\xi_n) - U_{n-1}(\xi_n)) \tag{9}$$

Consider the balancing between the highest nonlinear term $\beta U_n^2(U_{n+1} - U_{n+1})$ with the highest derivative term $\lambda \frac{\partial U_n}{\partial \xi_n}$ according to case(2) mentioned above, we have $M = 2$. We assume the solution of Eq. (11) can be expressed as

$$U_n(\xi_n) = a_0 + a_1 \psi_n(\xi_n) + a_2 \psi_n^2(\xi_n) + \frac{b_1}{\psi_n(\xi_n)} + \frac{b_2}{\psi^2(\xi_n)}, \tag{10}$$

where a_0, a_1, a_2, b_1 and b_2 are to be determined later, ψ_n satisfy the system of nonlinear Eqs. (1)–(3). With the aid of the expressions of U_n, U_{n+1}, U_{n-1} in Eq. (9), we have a nonlinear DDE which respect to ψ_n, ψ_{n+1}, ψ_{n-1}, we reduce the difference terms in Eq. (11) by using Eqs. (1) and (2) and reduce the differential terms by Eq. (3). We get an expression in rational polynomial form of ψ_n. Collecting the coefficients of $\psi_n^i, i = 0, 1, 2, \ldots\ldots$ and setting to zero yield an algebraic equation system for a_0, a_1, a_2, b_1, b_2 and λ. Solving this system of equations by Maple, we have three kinds of coefficients as

$$a_2 = 0,\ a_0 = -\frac{\alpha}{2\beta},\ \lambda = -\frac{A(\alpha^2 - 4\beta)}{8\beta},$$
$$a_1 = \pm\frac{A\sqrt{\alpha^2 - 4\beta}\delta}{4\beta},\ b_2 = 0,\ b_1 = \frac{\sqrt{\alpha^2 - 4\beta}}{4\beta A\delta} \tag{11}$$

Family(1). For $\delta = 1$, $\epsilon = 1$, $A = tanh(k)$, $\psi_n = tanh(\xi_n)$ or $coth(\xi_n)$, with Eq. (1), admits to solitary wave solutions of Eq. (9) as follows

$$u_n(\xi_n) = -\frac{\alpha}{2\beta} \pm \frac{tanh(k)\sqrt{\alpha^2 - 4\beta}\delta}{4\beta} tanh(\xi_n) + \frac{\sqrt{\alpha^2 - 4\beta}}{\beta tanh(k) tanh(\xi_n)}, \tag{12}$$

$$u_n(\xi_n) = -\frac{\alpha}{2\beta} \pm \frac{tanh(k)\sqrt{\alpha^2 - 4\beta}\delta}{4\beta} coth(\xi_n) + \frac{\sqrt{\alpha^2 - 4\beta}}{\beta tanh(k) coth(\xi_n)}, \tag{13}$$

Family(2). For $\delta = -1$, $\epsilon = 1$, $A = tan(k)$, $\psi_n = tan(\xi_n)$ or $-cot(\xi_n)$, with Eq. (13), admits to triangular function solutions of Eq. (9) as

$$u_n(\xi_n) = -\frac{\alpha}{2\beta} \pm \frac{tan(k)\sqrt{\alpha^2 - 4\beta}\delta}{4\beta} tan(\xi_n) + \frac{\sqrt{\alpha^2 - 4\beta}}{\beta tan(k) tan(\xi_n)}, \tag{14}$$

$$u_n(\xi_n) = -\frac{\alpha}{2\beta} - \frac{tan(k)\sqrt{\alpha^2 - 4\beta}\delta}{4\beta} cot(\xi_n) - \frac{\sqrt{\alpha^2 - 4\beta}}{\beta tan(k) cot(\xi_n)}, \tag{15}$$

Family(3). For $\delta = 1$, $\epsilon = 0$, $A = k$, $\psi_n = \frac{1}{\xi_n}$, with Eq. (13), admits to rational solution of Eq. (9) as

$$u_n(\xi_n) = -\frac{\alpha}{2\beta} \pm \frac{k\sqrt{\alpha^2 - 4\beta}}{4\beta\xi_n} + \frac{\sqrt{\alpha^2 - 4\beta}\xi_n}{4\beta k}, \tag{16}$$

$$\xi_n = kn - \frac{A(\alpha^2 - 4\beta)}{8\beta} t + c \tag{17}$$

It is to be noted that the solutions obtained from Eqs. (14)–(18) are exactly the same with that obtained in Wang (2009) by setting $b_1 = b_2 = 0$.

3.2. Example(2). Relativistic Toda coupled nonlinear difference equation

A second instructive model is the Relativistic Toda coupled nonlinear difference equation (Baldwin et al., 2004),

$$\frac{\partial u_n}{\partial t} = (1 + \alpha u_n)(v_n - v_{n-1}), \tag{18}$$

$$\frac{\partial v_n}{\partial t} = v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}), \tag{19}$$

where α is constant. The Toda lattice difference Eqs. (20) and (21) describe vibrations in mass-spring lattices with an exponential interaction force. Using the travelling wave solution we obtain

$$u(n,t) = U(\xi_n), V(n,t) = V(\xi_n), \xi_n = kn + \lambda t + c, \tag{20}$$

where k, c and λ are constants. Then the discrete Relativistic Toda difference equation becomes

$$\lambda\frac{\partial U_n}{\partial \xi_n} = (1 + \alpha U_n(\xi_n))(V_n(\xi_n) - U_{n-1}(\xi_n)), \tag{21}$$

$$\lambda\frac{\partial v_n}{\partial \xi_n} = V_n(\xi_n)(U_{n+1}(\xi_n) - U_n(\xi_n) + \alpha V_{n+1}(\xi_n) - \alpha V_{n-1}(\xi_n)) \tag{22}$$

Consider the balancing between the highest nonlinear term with the highest derivative term. In Eqs. (23) and (24) according to case(2) mentioned above, we have $M = 1$. Therefore, we assume the solution of Eqs. (23) and (24) can be expressed as

$$U_n(\xi_n) = a_0 + a_1\psi_n(\xi_n) + \frac{b_1}{\psi_n(\xi_n)}, \tag{23}$$

$$V_n(\xi_n) = c_0 + c_1\psi_n(\xi_n) + \frac{d_1}{\psi_n(\xi_n)}, \tag{24}$$

where a_0, a_1, b_1, c_0, c_1, *and* d_1, are to be determined later, ψ_n satisfy the system of nonlinear Eqs. (1)–(3). Substituting the expressions of U_n, U_{n+1}, U_{n-1} in Eqs. (23) and (24), we have a nonlinear DDE which respect to ψ_n, ψ_{n+1}, ψ_{n-1}. Collecting the coefficients of $\psi_n^i, i = 0, 1, 2, \ldots\ldots$ and setting to zero yield an algebraic equation system for a_0, a_1, b_1, c_0, c_1, d_1 and λ. Solving this system of equations, we have two different cases for coefficients as follows

Case(1)

$$b_1 = \lambda\epsilon, \alpha = -\frac{\lambda\delta}{c_1}, a_1 = \delta\lambda, c_1 = c_1, \lambda = \lambda, d_1 = \frac{\epsilon c_1}{\delta},$$
$$c_0 = -\frac{c_1(1 + \epsilon A^2\delta)}{A\delta}, a_0 = -\frac{\lambda^2 A^2\epsilon\delta^2 + \lambda^2\delta - Ac_1}{A\lambda\delta}, \tag{25}$$

Case(2)

$$b_1 = 0, \alpha = \frac{\lambda}{Ac_0}, a_1 = \delta\lambda, c_1 = -Ac_0\delta, \lambda = \lambda, d_1 = 0,$$
$$c_0 = c_0, a_0 = -\frac{\lambda^2 + A^2 c_0}{A\lambda}, \tag{26}$$

Case(3)

$$b_1 = \lambda\epsilon, a_1 = 0, c_1 = 0, \lambda = \lambda, d_1 = -Ac_0, c_0 = c_0,$$
$$\alpha = \frac{\lambda}{Ac_0}, a_0 = -\frac{\lambda^2 + A^2 c_0}{A\lambda}, \tag{27}$$

According to case(1), with $\delta = 1$, $\epsilon = 1$, $A = tanh(k)$, $\psi_n = tanh(\xi_n)$ or $coth(\xi_n)$, admits to solitary wave solutions of Eqs. (20) and (21) as

$$u_n(\xi_n) = -\frac{\lambda^2 tanh^2(k) + \lambda^2 - tanh(k)c_1}{tanh(k)\lambda} + \lambda tanh(\xi_n) + \frac{\lambda}{tanh(\xi_n)}, \tag{28}$$

$$v_n(\xi_n) = -\frac{c_1 sech^2(k)}{tanh(k)} + c_1 tanh(\xi_n) + \frac{c_1}{tanh(\xi_n)}, \tag{29}$$

$$u_n(\xi_n) = -\frac{\lambda^2 tanh^2(k) + \lambda^2 - tanh(k)c_1}{tanh(k)\lambda} + \lambda coth(\xi_n) + \frac{\lambda}{coth(\xi_n)}, \tag{30}$$

$$v_n(\xi_n) = -\frac{c_1 sech^2(k)}{tanh(k)} + c_1 coth(\xi_n) + \frac{c_1}{coth(\xi_n)}, \tag{31}$$

In view of case(1), with $\delta = -1$, $\epsilon = 1$, $A = tan(k)$, $\psi_n = tan(\xi_n)$ or $-cot(\xi_n)$, admits to triangular function solutions of Eqs. (20) and (21) as

$$u_n(\xi_n) = \frac{\lambda^2 tan^2(k) - \lambda^2 - tan(k)c_1}{tan(k)\lambda} - \lambda tan(\xi_n) + \frac{\lambda}{tan(\xi_n)}, \quad (32)$$

$$v_n(\xi_n) = \frac{c_1(1 - tan^2(k))}{tan(k)} + c_1 tan(\xi_n) - \frac{c_1}{tan(\xi_n)}, \quad (33)$$

$$u_n(\xi_n) = \frac{\lambda^2 tan^2(k) - \lambda^2 - tan(k)c_1}{tan(k)\lambda} + \lambda cot(\xi_n) + \frac{\lambda}{cot(\xi_n)}, \quad (34)$$

$$v_n(\xi_n) = \frac{c_1(1 - tan^2(k))}{tan(k)} - c_1 cot(\xi_n) + \frac{c_1}{cot(\xi_n)}, \quad (35)$$

Using case(1), when $\delta = 1,\ \epsilon = 0,\ A = k,\ \psi_n = \frac{1}{\xi_n}$, we have a rational solution as

$$u_n(\xi_n) = -\frac{\lambda^2 - kc_1}{k\lambda} + \frac{\lambda}{\xi_n}, \quad (36)$$

$$v_n(\xi_n) = -\frac{c_1}{k} + \frac{c_1}{\xi_n}, \quad (37)$$

$$\xi_n = kn + \lambda t + c \quad (38)$$

According to case(2), with $\delta = 1,\ \epsilon = 1,\ A = tanh(k),\ \psi_n = tanh(\xi_n)$ or $coth(\xi_n)$, admits to solitary wave solutions of Eqs. (20) and (21) taken as

$$u_n(\xi_n) = -\frac{\lambda^2 + tanh^2(k)c_0}{tanh(k)\lambda} + \lambda tanh(\xi_n), \quad (39)$$

$$v_n(\xi_n) = c_0[1 - tanh(k)tanh(\xi_n)], \quad (40)$$

$$u_n(\xi_n) = -\frac{\lambda^2 + tanh^2(k)c_0}{tanh(k)\lambda} + \lambda coth(\xi_n), \quad (41)$$

$$v_n(\xi_n) = c_0[1 - tanh(k)coth(\xi_n)], \quad (42)$$

In view of case(2), when $\delta = -1,\ \epsilon = 1,\ A = tan(k),\ \psi_n = tan(\xi_n)$ or $-cot(\xi_n)$, admits to triangular function solutions of Eqs. (20) and (21) as

$$u_n(\xi_n) = -\frac{\lambda^2 + tan^2(k)c_0}{tan(k)\lambda} + \lambda tan(\xi_n), \quad (43)$$

$$v_n(\xi_n) = c_0[1 - tan(k)tan(\xi_n)], \quad (44)$$

$$u_n(\xi_n) = -\frac{\lambda^2 + tan^2(k)c_0}{tan(k)\lambda} - \lambda cot(\xi_n), \quad (45)$$

$$v_n(\xi_n) = c_0[1 + tan(k)cot(\xi_n)] \quad (46)$$

Form case(2), when $\delta = 1,\ \epsilon = 0,\ A = k,\ \psi_n = \frac{1}{\xi_n}$, we have a rational solution as

$$u_n(\xi_n) = -\frac{\lambda^2 + k^2 c_0}{k\lambda} + \frac{\lambda}{\xi_n}, \quad (47)$$

$$v_n(\xi_n) = c_0 - \frac{kc_0}{\xi_n}, \quad (48)$$

$$\xi_n = kn + \lambda t + c \quad (49)$$

It is to be noted that the solutions obtained from Eqs. (30)–(50) are quite good with that obtained in Baldwin et al. (2004). For simplicity case(3) is omitted here.

3.3. Example(3). Toda lattice nonlinear differential difference equation

In this case, we consider the Toda lattice nonlinear differential difference equation (Baldwin et al., 2004),

$$\frac{\partial u_n}{\partial t} = u_n(v_n - v_{n-1}), \quad (50)$$

$$\frac{\partial v_n}{\partial t} = v_n(u_{n+1} - u_n) \quad (51)$$

To look for the travelling wave solutions of Eqs. (52) and (53), we use the transformation

$$u(n,t) = U(\xi_n), V(n,t) = V(\xi_n), \xi_n = kn + \lambda t + c, \quad (52)$$

where $k,\ c$ and λ are constants. Then the discrete Toda difference equation reduces to

$$\lambda\frac{\partial U_n}{\partial \xi_n} = U_n(\xi_n)(V_n(\xi_n) - U_{n-1}(\xi_n)), \quad (53)$$

$$\lambda\frac{\partial v_n}{\partial \xi_n} = V_n\xi_n(U_{n+1}(\xi_n) - U_n(\xi_n)) \quad (54)$$

Consider the balancing between the highest nonlinear term and the highest derivative term in Eqs. (53) and (54) according to case(2) mentioned above, we have $M = 1$. Therefore, we assume the solution of Eqs. (55) and (56) can be expressed as

$$U_n(\xi_n) = a_0 + a_1\psi_n(\xi_n) + \frac{b_1}{\psi_n(\xi_n)}, \quad (55)$$

$$V_n(\xi_n) = c_0 + c_1\psi_n(\xi_n) + \frac{d_1}{\psi_n(\xi_n)}, \quad (56)$$

where $a_0,\ a_1,\ b_1,\ c_0,\ c_1,\ d_1$, are to be determined later, ψ_n satisfy the system of nonlinear Eqs. (1) and (2). Substituting the expressions of $U_n,\ U_{n+1}, U_{n-1}$ in Eqs. (55) and (56), we have a nonlinear DDE which respect to $\psi_n,\ \psi_{n+1},\ \psi_{n-1}$. Collecting the coefficients of $\psi_n^i, i = 0, 1, 2, \ldots\ldots$ and setting to zero yield an algebraic equation system for $a_0,\ a_1,\ b_1,\ c_0,\ c_1,\ d_1$ and λ. Solving this system of equations, we obtain

Case(1)

$$b_1 = -\frac{c_1\epsilon}{\delta}, a_1 = -c_1, c_1 = c_1, \lambda = -\frac{c_1}{\delta}, d_1 = \frac{\epsilon c_1}{\delta},$$
$$c_0 = \frac{c_1(1 + \epsilon A^2\delta)}{A\delta}, a_0 = \frac{c_1(1 + \delta A^2\epsilon)}{\delta A} \quad (57)$$

Case(2)

$$b_1 = -A\epsilon c_0, a_1 = 0, c_1 = 0, \lambda = -Ac_0, d_1 = Ac_0\epsilon,$$
$$c_0 = c_0, a_0 = c_0 \quad (58)$$

According to case(1), with $\delta = 1,\ \epsilon = 1,\ A = tanh(k),\ \psi_n = tanh(\xi_n)$ or $coth(\xi_n)$, admits to solitary wave solutions of Eqs. (52) and (53) as

$$u_n(\xi_n) = \frac{c_1 sech^2(k)}{tanh(k)} - c_1 tanh(\xi_n) - \frac{c_1}{tanh(\xi_n)}, \quad (59)$$

$$v_n(\xi_n) = \frac{c_1 sech^2(k)}{tanh(k)} + c_1 tanh(\xi_n) + \frac{c_1}{tanh(\xi_n)}, \quad (60)$$

$$u_n(\xi_n) = \frac{c_1 sech^2(k)}{tanh(k)} - c_1 coth(\xi_n) - \frac{c_1}{coth(\xi_n)}, \quad (61)$$

$$v_n(\xi_n) = \frac{c_1 sech^2(k)}{tanh(k)} + c_1 coth(\xi_n) + \frac{c_1}{coth(\xi_n)} \quad (62)$$

In view of case(1), when $\delta = -1,\ \epsilon = 1,\ A = tan(k)$, $\psi_n = tan(\xi_n)$ or $-cot(\xi_n)$, admits to triangular function solutions of Eqs. (52) and (53) as

$$u_n(\xi_n) = \frac{c_1(1 - tan^2(k))}{tan(k)} - c_1 tan(\xi_n) - \frac{c_1}{tan(\xi_n)}, \tag{63}$$

$$v_n(\xi_n) = -\frac{c_1(1 - tan^2(k))}{tan(k)} + c_1 tan(\xi_n) - \frac{c_1}{tan(\xi_n)}, \tag{64}$$

$$u_n(\xi_n) = -\frac{c_1(1 - tan^2(k))}{tan(k)} + c_1 cot(\xi_n) + \frac{c_1}{cot(\xi_n)}, \tag{65}$$

$$v_n(\xi_n) = \frac{c_1(1 - tan^2(k))}{tan(k)} - c_1 cot(\xi_n) - \frac{c_1}{cot(\xi_n)} \tag{66}$$

Using case(1), when $\delta = 1,\ \epsilon = 0,\ A = k,\ \psi_n = \frac{1}{\xi_n}$, we have a rational solution as

$$u_n(\xi_n) = \frac{c_1}{k} - \frac{c_1}{\xi_n}, \tag{67}$$

$$v_n(\xi_n) = \frac{c_1}{k} + \frac{c_1}{\xi_n}, \tag{68}$$

$$\xi_n = kn - \frac{c_1}{\delta} t + c \tag{69}$$

According to case(2), with $\delta = 1,\ \epsilon = 1,\ A = tanh(k)$, $\psi_n = tanh(\xi_n)$ or $coth(\xi_n)$, admits to solitary wave solutions as

$$u_n(\xi_n) = c_0 - \frac{c_0 tanh(k)}{tanh(\xi_n)}, \tag{70}$$

$$v_n(\xi_n) = c_0 + \frac{c_0 tanh(k)}{tanh(\xi_n)}, \tag{71}$$

$$u_n(\xi_n) = c_0 - \frac{c_0 tanh(k)}{coth(\xi_n)}, \tag{72}$$

$$v_n(\xi_n) = c_0 + \frac{c_0 tanh(k)}{coth(\xi_n)} \tag{73}$$

By means of case(2), when $\delta = -1,\ \epsilon = 1,\ A = tan(k)$, $\psi_n = tan(\xi_n)$ or $-cot(\xi_n)$, admits to triangular function solutions as

$$u_n(\xi_n) = c_0 - \frac{c_0 tan(k)}{tan(\xi_n)}, \tag{74}$$

$$v_n(\xi_n) = c_0 + \frac{c_0 tan(k)}{tan(\xi_n)}, \tag{75}$$

$$u_n(\xi_n) = c_0 + \frac{c_0 tan(k)}{cot(\xi_n)}, \tag{76}$$

$$v_n(\xi_n) = c_0 - \frac{c_0 tan(k)}{cot(\xi_n)}, \tag{77}$$

$$\xi_n = kn - Ac_0 t + c \tag{78}$$

The solutions obtained Eqs. (59)–(78) are quite good with that obtained in Baldwin et al. (2004).

4. Conclusions and discussion

In summary, the discrete tanh method (Wang, 2009) is used for constructing exact solutions to nonlinear difference-differential equations (DDEs) arising in mathematical physics,namely,Hybrid equation, Toda lattice equation and Relativistic Toda lattice difference equations.

As a result, we obtain many kinds of exact solutions including soliton solutions,periodic solutions and rational solutions in a uniform way if solutions of these kinds exist.

Here, we presented a generalized discrete tanh method based on the general ansatz (8) in which the exponent of function may take both positive and negative values on the contrary to the solution ansatz (7) where its exponent is only positive values.

Finally, it is worth noting that the new solutions obtained by the proposed method confirm the correctness of those obtained by other methods. The method is straightforward and concise, and it can also be applied to other nonlinear difference differential equations in mathematical physics. This is our task in future work.

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