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حلول دقيقة لمعادلة كلين-جوردن (K-G) التكعيبية ذات البعد (2+1) ومعادلة زاخرووف-كوزنتسوف (Z-K) ذات البعد (3+1) باستخدام طريقة المعادلة البسيطة المعدلة

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Originated

الملخص:

الحلول الدقيقة لمعادلات التطور اللاخطية (NIEEs) تلعب دور حيوي في الكشف عن الالية الداخلية للظواهر الفيزيائية المعقدة. في هذا البحث تم تنفيذ طريقة المعادلة البسيطة المعدلة (MSE) لإيجاد حلول دقيقة لمعادلات التطور اللاخطية بواسطة معادلة كلين-جوردن (K-G) التكعيبية ذات بعد (1+2) ومعادلة زاخرووف-كوزنتسوف (Z-K) ذات بعد (3+1) وقد تم تحقيق الحلول الدقيقة والتي تتضمن المعاملات. ولعدة قيم خاصة للمعاملات، فان حلول الموجات المنفردة (Solitons) قد نشأت من الحلول الدقيقة. لقد تم في هذا البحث التأكد بأن طريقة المعادلة البسيطة المعدلة (MSE) تعتبر ايضا اداة رياضية يمكن استخدمها لإيجاد حلول مضبوطة لمعادلات التطور اللاخطية في مجال الفيزياء الرياضية.

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ORIGINAL ARTICLE

Exact solutions of the (2 + 1)-dimensional cubic Klein–Gordon equation and the (3 + 1)-dimensional Zakharov–Kuznetsov equation using the modified simple equation method

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Abstract Exact solutions of nonlinear evolution equations (NLEEs) play a vital role to reveal the internal mechanism of complex physical phenomena. In this article, we implemented the modified simple equation (MSE) method for finding the exact solutions of NLEEs via the (2 + 1)-dimensional cubic Klein–Gordon (cKG) equation and the (3 + 1)-dimensional Zakharov–Kuznetsov (ZK) equation and achieve exact solutions involving parameters. When the parameters are assigned special values, solitary wave solutions are originated from the exact solutions. It is established that the MSE method offers a further influential mathematical tool for constructing exact solutions of NLEEs in mathematical physics.

1. Introduction

Nonlinear phenomena exist in all areas of science and engineering, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and so on. It is well known that many NLEEs are widely used to describe these complex physical phenomena. Therefore, research to look for exact solutions of NLEEs is extremely crucial. So, to find effective methods to discover analytic and numerical solutions of nonlinear equations have drawn an abundance of interest by a diverse group of researchers. Accordingly, they established many powerful and efficient methods and techniques to explore the exact traveling wave solutions of nonlinear physical phenomena, such as, the Hirota's bilinear transformation method (Hirota, 1973; Hirota and Satsuma, 1981), the tanh-function method (Malfliet, 1992; Nassar et al., 2011), the (G'/G)-expansion method (Wang et al., 2008; Zayed, 2010; Zayed and Gepreel, 2009; Akbar et al., 2012a,b,c,d; Akbar and Ali, 2011a; Shehata, 2010), the

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Exp-function method (He and Wu, 2006; Akbar and Ali, 2011b; Naher et al., 2011, 2012), the homogeneous balance method (Wang, 1995; Zayed et al., 2004), the F-expansion method (Zhou et al., 2003), the Adomian decomposition method (Adomian, 1994), the homotopy perturbation method (Mohiud-Din, 2007), the extended tanh-method (Abdou, 2007; Fan, 2000), the auxiliary equation method (Sirendaoreji, 2004), the Jacobi elliptic function method (Ali, 2011), Weierstrass elliptic function method (Liang et al., 2011), modified Exp-function method (He et al., 2012), the modified simple equation method (Jawad et al., 2010; Zayed, 2011a, Zayed and Ibrahim, 2012; Zayed and Arnous, 2012), the extended multiple Riccati equations expansion method (Gepreel and Shehata, 2012; Gepreel, 2011a; Zayed and Gepreel, 2011) and others (Gepreel, 2011b,c).

The objective of this article is to look for new study relating to the MSE method to examine exact solutions to the celebrated (2+1)-dimensional cKG equation and the (3+1)-dimensional ZK equations to establish the advantages and effectiveness of the method. The cKG equation is used to model many different nonlinear phenomena, including the propagation of dislocation in crystals and the behavior of elementary particles and the propagation of fluxions in Josephson junctions. The (3+1)-dimensional Zakharov–Kuznetsov equation describes weakly nonlinear wave process in dispersive and isotropic media e.g., waves in magnetized plasma or water waves in shear flows.

The article is organized as follows: In Section 2, the MSE method is discussed. In Section 3 we exert this method to the nonlinear evolution equations pointed out above, in Section 4 physical explanation, in Section 5 comparisons and in Section 6 conclusions are given.

2. The MSE method

Suppose the nonlinear evolution equation is in the form,

$$F(u, u_t, u_x, u_y, u_z, u_{xx}, u_{tt}, \ldots) = 0, \tag{2.1}$$

where F is a polynomial of $u(x, y, z, t)$ and its partial derivatives wherein the highest order derivatives and nonlinear terms are involved. The main steps of the MSE method (Jawad et al., 2010; Zayed, 2011a; Zayed and Ibrahim, 2012; Zayed and Arnous, 2012) are as follows:

Step 1: The traveling wave transformation,

$$u(x, y, z, t) = u(\xi), \qquad \xi = x + y + z \pm \lambda t \tag{2.2}$$

allows us to reduce Eq. (2.1) into the following ordinary differential equation (ODE):

$$P(u, u\prime, u\prime\prime, \ldots) = 0, \tag{2.3}$$

where P is a polynomial in $u(\xi)$ and its derivatives, while $u\prime(\xi) = \frac{du}{d\xi}$.

Step 2: We suppose that Eq. (2.3) has the solution in the form

$$u(\xi) = \sum_{i=0}^{n} C_i \left(\frac{\phi\prime(\xi)}{\phi(\xi)}\right)^i, \tag{2.4}$$

where $C_i (i = 0, 1, 2, 3, \ldots)$ are arbitrary constants to be determined, such that $C_n \neq 0$ and $\phi(\xi)$ is an unspecified function to be found out afterward.

Step 3: We determine the positive integer n appearing in Eq. (2.4) by considering the homogeneous balance between the highest order derivatives and the highest order nonlinear terms come out in Eq. (2.3).

Step 4: We substitute Eq. (2.4) into (2.3) and then we account the function $\phi(\xi)$. As a result of this substitution, we get a polynomial of $(\phi'(\xi)/\phi(\xi))$ and its derivatives. In this polynomial, we equate the coefficients of same power of $\phi^{-j}(\xi)$ to zero, where $j \geqslant 0$. This procedure yields a system of equations which can be solved to find $C_i, \phi(\xi)$ and $\phi\prime(\xi)$. Then the substitution of the values of C_i, $\phi(\xi)$ and $\phi'(\xi)$ into Eq. (2.4) completes the determination of exact solutions of Eq. (2.1).

3. Applications

3.1. The (2+1)-dimensional cubic Klein–Gordon (cKG) equation

In this sub-section, first we will exert the MSE method to find the exact solutions and solitary wave solutions of the celebrated (2+1)-dimensional cKG equation,

$$u_{xx} + u_{yy} - u_{tt} + \alpha u + \beta u^3 = 0 \tag{3.1}$$

where α and β are non zero constants.

The traveling wave transformation

$$u = u(x, y, t), \quad \xi = x + y - \lambda t, \quad u(x, y, t) = u(\xi), \tag{3.2}$$

transforms the Eq. (3.1) to the following ODE:

$$(2 - \lambda^2)u\prime\prime + \alpha u + \beta u^3 = 0. \tag{3.3}$$

Balancing the highest order derivative and nonlinear term of the highest order, yields $n = 1$.

Thus, the solution Eq. (2.4) takes the form,

$$u(\xi) = C_0 + C_1 \left(\frac{\phi\prime(\xi)}{\phi(\xi)}\right), \tag{3.4}$$

where C_0 and C_1 are constants such that $C_1 \neq 0$, and $\phi(\xi)$ is an unspecified function to be determined. It is simple to calculate that

$$u\prime = C_1 \left(\frac{\phi\prime\prime}{\phi} - \left(\frac{\phi\prime}{\phi}\right)^2\right), \tag{3.5}$$

$$u\prime\prime = C_1 \left(\frac{\phi\prime\prime\prime}{\phi}\right) - 3C_1 \left(\frac{\phi\prime\prime\phi\prime}{\phi^2}\right) + 2C_1 \left(\frac{\phi\prime}{\phi}\right)^3, \tag{3.6}$$

$$u^3 = C_1^3 \left(\frac{\phi\prime}{\phi}\right)^3 + 3C_1^2 C_0 \left(\frac{\phi\prime}{\phi}\right)^2 + 3C_1 C_0^2 \left(\frac{\phi\prime}{\phi}\right) + C_0^3. \tag{3.7}$$

Substituting the values of u, u'', and u^3 into Eq. (3.3) and equating the coefficients of $\phi^0, \phi^{-1}, \phi^{-2}, \phi^{-3}$ to zero, yields

$$\alpha C_0 + \beta C_0^3 = 0. \tag{3.8}$$

$$(\lambda^2 - 2)\phi''' - \left(\alpha + 3\beta C_0^2\right)\phi\prime = 0, \tag{3.9}$$

$$(\lambda^2 - 2)\phi\prime\prime + \beta C_0 C_1 \phi' = 0, \tag{3.10}$$

$$\left(\beta C_1^3 - 2(\lambda^2 - 2)C_1\right)(\phi')^3 = 0. \tag{3.11}$$

Solving Eq. (3.8), we obtain

$$C_0 = 0, \pm I\sqrt{\left(\frac{\alpha}{\beta}\right)}.$$

And solving Eq. (3.11), we obtain

$$C_1 = \pm\sqrt{\left(\frac{2(\lambda^2-2)}{\beta}\right)}, \text{ since } C_1 \neq 0.$$

Case-I: When $C_0 = 0$, we obtain trivial solution, therefore the case is rejected.

Case-II: When $C_0 = \pm I\sqrt{\left(\frac{\alpha}{\beta}\right)}$, Eqs. (3.9) and (3.10) yield

$$\frac{\phi'''}{\phi''} + l = 0, \tag{3.12}$$

where $l = \frac{\alpha+3\beta C_0^2}{\beta C_0 C_1}$

Integrating, Eq. (3.12) with respect to ξ, we obtain

$$\phi''(\xi) = a_1 \exp(-l\xi) \tag{3.13}$$

Using Eq. (3.13), from Eq. (3.10), we obtain

$$\phi'(\xi) = \frac{-a_1(\lambda^2-2)\exp(-l\xi)}{\beta C_0 C_1}. \tag{3.14}$$

Upon integration, we obtain

$$\phi(\xi) = a_2 + \frac{a_1(\lambda^2-2)\exp(-l\xi)}{\alpha + 3\beta C_0^2}. \tag{3.15}$$

where a_1 and a_2 are constants of integration. Therefore, the exact solution of the cKG Eq. (3.1) is

$$u(\xi) = C_0 - \frac{\left(\alpha+3\beta C_0^2\right)}{\beta C_0} \times \frac{a_1(\lambda^2-2)\exp(-l\xi)}{a_2\left(\alpha+3\beta C_0^2\right) + a_1(\lambda^2-2)\exp(-l\xi)}. \tag{3.16}$$

Substituting the values of C_0, C_1, and l and simplifying, we obtain

$$u(\xi) = \pm I \times \sqrt{\left(\frac{\alpha}{\beta}\right)}\left(1 - \frac{2a_1(\lambda^2-2)\left\{\cosh\left(\sqrt{\left(\frac{\alpha}{-2(\lambda^2-2)}\right)}\xi\right) + \sinh\left(\sqrt{\left(\frac{\alpha}{-2(\lambda^2-2)}\right)}\xi\right)\right\}}{(a_1(\lambda^2-2) - 2a_2\alpha)\cosh\left(\sqrt{\left(\frac{\alpha}{-2(\lambda^2-2)}\right)}\xi\right) + (a_1(\lambda^2-2) + 2a_2\alpha)\sinh\left(\sqrt{\left(\frac{\alpha}{-2(\lambda^2-2)}\right)}\xi\right)}\right), \tag{3.17}$$

where $\xi = x + y - \lambda t$.

We can arbitrarily choose the parameters a_1 and a_2. Therefore, if we set $a_1 = \frac{2a_2\alpha}{\lambda^2-2}$, Eq. (3.17) reduces to:

$$u_{1,2}(x,y,t) = \pm I\sqrt{\left(\frac{\alpha}{\beta}\right)} \times \coth\left(\sqrt{\left(\frac{\alpha}{-2(\lambda^2-2)}\right)}(x+y-\lambda t)\right). \tag{3.18}$$

Again if we set $a_1 = -\frac{2a_2\alpha}{\lambda^2-2}$, Eq. (3.17) reduces to:

$$u_{3,4}(x,y,t) = \pm I\sqrt{\left(\frac{\alpha}{\beta}\right)} \times \tanh\left(\sqrt{\left(\frac{\alpha}{-2(\lambda^2-2)}\right)}(x+y-\lambda t)\right). \tag{3.19}$$

Using hyperbolic function identities, from Eqs. (3.18) and (3.19), we obtain the following periodic solutions

$$u_{5,6}(x,y,t) = \pm\sqrt{\left(\frac{\alpha}{\beta}\right)} \times \cot\left(\sqrt{\left(\frac{\alpha}{2(\lambda^2-2)}\right)}(x+y-\lambda t)\right). \tag{3.20}$$

And

$$u_{7,8}(x,y,t) = \pm\sqrt{\left(\frac{\alpha}{\beta}\right)} \times \tan\left(\sqrt{\left(\frac{\alpha}{2(\lambda^2-2)}\right)}(x+y-\lambda t)\right). \tag{3.21}$$

Remark 1. Solutions (3.16)–(3.21) have been verified by substituting them back into the original equation and found correct.

3.2. The (3+1)-dimensional Zakharov–Kuznetsov (ZK) equation

Now we will study the MSE method to find exact solutions and then the solitary wave solutions to the (3+1)-dimensional ZK equation

$$u_t + auu_x + u_{xx} + u_{yy} + u_{zz} = 0, \tag{3.22}$$

where

$$u = u(x,y,z,t), \xi = x + y + z - \lambda t, u(x,y,z,t) = u(\xi). \tag{3.23}$$

The traveling wave transformation (3.23) reduces to Eq. (3.22) to the following ODE:

$$-\lambda u' + auu' + 3u'' = 0. \tag{3.24}$$

Integrating Eq. (3.24) with respect to ξ, we obtain

$$-\lambda u + \frac{1}{2}au^2 + 3u' = 0. \tag{3.25}$$

Balancing the highest order derivative u' and nonlinear term u^2, we obtain $2n = n + 1$, which gives $n = 1$.

Therefore, the solution (2.4) takes the form

$$u(\xi) = C_0 + C_1\left(\frac{\phi'}{\phi}\right), \tag{3.26}$$

where C_0 and C_1 are constants such that $C_1 \neq 0$, and $\phi(\xi)$ is an unstipulated function to be determined. It is easy to make out that

$$u' = C_1\left(\frac{\phi''}{\phi} - \left(\frac{\phi'}{\phi}\right)^2\right), \tag{3.27}$$

$$u^2 = C_0^2 + 2C_0C_1\left(\frac{\phi'}{\phi}\right) + C_1^2\left(\frac{\phi'}{\phi}\right)^2. \tag{3.28}$$

Substituting the values of u, u' and u^2 from (3.26), (3.27), and (3.28) into Eq. (3.25) and then equating the coefficients of ϕ^0, ϕ^{-1}, and ϕ^{-2} to zero, we respectively obtain

$$-\lambda C_0 + \frac{1}{2}aC_0^2 = 0, \tag{3.29}$$

$$3C_1\phi'' + C_1(aC_0 - \lambda)\phi' = 0, \tag{3.30}$$

$$\left(\frac{1}{2}aC_1^2 - 3C_1\right)(\phi')^2 = 0. \tag{3.31}$$

From Eq. (3.29), we obtain

$$C_0 = 0, \frac{2\lambda}{a}.$$

And from Eq. (3.31), we obtain

$$C_1 = \frac{6}{a}, \text{ since } C_1 \neq 0.$$

Case-I: When $C_0 = 0$ Eqs. (3.30) yield

$$\frac{\phi''}{\phi'} = \frac{\lambda}{3}. \tag{3.32}$$

Integrating Eq. (3.32), we obtain

$$\phi'(\xi) = a_1 \exp\left(\frac{\lambda}{3}\xi\right), \tag{3.33}$$

where a_1 is a constant of integration.

Integrating (3.33) with respect to ξ, we obtain

$$\phi(\xi) = a_2 + \frac{3a_1}{\lambda}\exp\left(\frac{\lambda}{3}\xi\right), \tag{3.34}$$

where a_2 is a constant of integration.

Therefore, the exact solution of the (3 + 1)-dimensional ZK equation (3.22) is

$$u(\xi) = C_0 + C_1\left(\frac{\lambda a_1 \exp\left(\frac{\lambda}{3}\xi\right)}{\lambda a_2 + 3a_1 \exp\left(\frac{\lambda}{3}\xi\right)}\right). \tag{3.35}$$

Substituting the values of C_0 and C_1 into Eq. (3.35) and simplifying, we obtain

$$u(\xi) = \frac{2}{a}\left(\frac{3\lambda a_1\left(\cosh\left(\frac{\lambda}{6}\xi\right) + \sinh\left(\frac{\lambda}{6}\xi\right)\right)}{(3a_1 + \lambda a_2)\cosh\left(\frac{\lambda}{6}\xi\right) + (3a_1 - \lambda a_2)\sinh\left(\frac{\lambda}{6}\xi\right)}\right), \tag{3.36}$$

where $\xi = x + y + z - \lambda t$.

We can randomly choose the parameters a_1 and a_2. Setting $a_1 = \frac{\lambda a_2}{3}$, Eq. (3.36) reduces to:

$$u_1(x, y, z, t) = \frac{\lambda}{a}\left(1 + \tanh\left(\frac{\lambda}{6}(x + y + z - \lambda t)\right)\right). \tag{3.37}$$

Again setting $a_1 = -\frac{\lambda a_2}{3}$ Eq. (3.36) reduces to:

$$u_2(x, y, z, t) = \frac{\lambda}{a}\left(1 + \coth\left(\frac{\lambda}{6}(x + y + z - \lambda t)\right)\right). \tag{3.38}$$

Case-II: When $C_0 = \frac{2\lambda}{a}$, from Eq. (3.30), executing the parallel course of action which described in Case-I, we obtain

$$u_3(\xi) = \frac{2\lambda}{a}\left(1 + \frac{3a_1\left(1 - \tanh\left(\frac{\lambda}{3}\xi\right)\right)}{\lambda a_2 sech\left(\frac{\lambda}{3}\xi\right) - 3a_1\left(1 - \tanh\left(\frac{\lambda}{3}\xi\right)\right)}\right), \tag{3.39}$$

where $\xi = x + y + z - \lambda t$.

Remark 2. Solutions 3.35, (3.36)–(3.39) have been checked by substituting them back into the original equation and found correct.

4. Physical explanations

In this section, physical explanations of the determined solutions are illustrated to show the effectiveness and convenience of the MSE method for seeking exact solitary wave solution and periodic traveling wave solution of nonlinear wave equations named (2 + 1)-dimensional cKG equation and the (3 + 1)-dimensional ZK equation.

4.1. Explanations

4.1.1. The (2 + 1)-dimensional cubic Klein–Gordon (cKG) equation

The wave speed λ and the nonzero constants α, and β play an important role in the physical structure of the solutions obtained in Eqs. (3.18)–(3.21). Now we will discuss the physical structure of Eqs. (3.18) and (3.19). For different values of λ, α and β the following cases arise in Eqs. (3.18) and (3.19):

(i) In Eqs. (3.18)–(3.21), the wave speed $\lambda \neq \pm\sqrt{(2)}$ or $\lambda \neq 0$.
(ii) If $-\sqrt{2} < \lambda < \sqrt{2}$ and $\alpha > 0$, $\beta > 0$ then from Eqs. (3.18) and (3.19) we get complex soliton solutions.
(iii) If $-\sqrt{2} < \lambda < \sqrt{2}$ and $\alpha < 0$, $\beta < 0$ then Eqs. (3.18) and (3.19) turn into Eqs. (3.20) and (3.21) respectively, i.e., they provide plane periodic solutions.
(iv) If $-\sqrt{2} < \lambda < \sqrt{2}$ and $\alpha < 0$, $\beta > 0$ then Eqs. (3.18) and (3.19) give complex periodic solutions.
(v) If $-\sqrt{2} < \lambda < \sqrt{2}$ and $\alpha > 0$, $\beta < 0$ then Eq. (3.18) fabricates singular kink solution and Eq. (3.19) furnishes kink solutions.
(vi) If $\lambda < -\sqrt{2}$ or $\lambda > \sqrt{2}$ and $\alpha > 0$, $\beta > 0$ then Eqs. (3.18) and (3.19) turn into Eqs. (3.20) and (3.21) respectively, i.e., they provide plane periodic solutions.
(vii) If $\lambda < -\sqrt{2}$ or $\lambda > \sqrt{2}$ and $\alpha < 0$, $\beta < 0$ then Eqs. (3.18) and (3.19) donate complex solitons.
(viii) If $\lambda < -\sqrt{2}$ or $\lambda > \sqrt{2}$ and $\alpha < 0$, $\beta > 0$ then Eq. (3.18) constructs singular kink solution and Eq. (3.19) assembles kink solutions.
(ix) If $\lambda < -\sqrt{2}$ or $\lambda > \sqrt{2}$ and $\alpha > 0$, $\beta < 0$ then Eqs. (3.18) and (3.19) represent complex periodic solutions.

4.1.2. The (3 + 1)-dimensional Zakharov–Kuznetsov (ZK) equation

The wave speed λ and the nonzero constant a play a significant role in the physical structure of the solutions obtained in Eqs. (3.37)–(3.39). For different values of λ and a the following cases arise in Eqs. (3.37)–(3.39):

(i) In Eqs. (3.37)–(3.39) $\lambda \neq 0$ and $a \neq 0$. For other values of λ and a Eq. (3.37) produce kink solutions and Eq. (3.38) construct singular kink solutions.
(ii) If $\lambda < 0$ and $a < 0$ or $a > 0$ then Eq. (3.39) provides singular soliton solutions.
(iii) If $\lambda > 0$ and $a > 0$ then Eq. (3.39) provides soliton solutions.
(iv) If $\lambda > 0$ and $a < 0$ then Eq. (3.39) provides kink wave solutions.
(v) In Eq. (3.39) $a_1, a_2 \neq 0$.

4.2. Graphical representation

Some of our obtained traveling wave solutions are represented in the following figures via symbolic computation software Maple with explanations:

The solution Eq. (3.18) that comes infinity as in trigonometry, is Singular kink solution. Fig. 1 shows the shape of the exact singular kink-type solution of the (2+1)-dimensional cKG Eq. (3.1) (only shows the shape of Eq. (3.18) with wave speed $\lambda = -1$, $y = 2, \alpha = 1$, $\beta = -1$ and $-10 \leqslant x,t \leqslant 10$).

The solution Eq. (3.19) is called the kink solution. Fig. 2 shows the shape of the exact kink-type solution of the (2+1)-dimensional cKG Eq. (3.1) (only shows the shape of Eq. (3.19) with wave speed $\lambda = 2$, $y = 0, \alpha = -1$, $\beta = 2$ and $-10 \leqslant x,t \leqslant 10$)). The disturbance represented by $u(x,y,t)$ is moving in the positive x-direction. If we take wave speed $\lambda < 0$, then the propagation will be in the negative x-direction.

Eqs. (3.20) and (3.21) are the exact periodic traveling wave solutions of the (2+1)-dimensional cKG Eq. (3.1), shapes are represented in Figs. 3 and 4, respectively. Fig. 3 shows the shape of Eq. (3.20) with wave speed $\lambda = 2$, $y = 2$, $\alpha = 0.50, \beta = 1$ in the interval $-10 \leqslant x,t \leqslant 10$ and Fig. 4 shows the shape of Eq. (3.21) with wave speed $\lambda = 2$, $y = 0, \alpha = 1, \beta = 2$ in the interval $-10 \leqslant x,t \leqslant 10$. The Eq. (3.20) is singular periodic traveling wave solution and Eq. (3.21) is periodic traveling wave solution.

The solution Eqs. (3.37) and (3.39) are the kink solutions. Figs. 5 and 7 show the shape of the exact kink-type solution of the (3+1)-dimensional ZK Eq. (3.22) (only show the shape of Eq. (3.37) with wave speed $\lambda = 2, a = 1$, $y = z = 0$ in the interval $-10 \leqslant x$, $t \leqslant 10$ and Eq. (3.39) with wave speed $\lambda = 1$, $a = -1$, $a_1 = 1$, $a_2 = -1$, $y = 1, z = 1$ in the interval $-10 \leqslant x$, $t \leqslant 10$ respectively. The disturbance with $y = z = 0$ represented by $u(x, y, z, t)$ is moving in the positive x-direction. If we take wave speed $\lambda < 0$ the propagation will be in the negative x-direction. These types of waves are traveling waves whose evolution from one asymptotical state at $\xi \to -\infty$ to other asymptotical state at $\xi \to \infty$.

The solution Eq. (3.38) that comes infinity, is singular kink wave solution. Fig. 6 shows the shape of the exact singular kink-type solution of the (3+1)-dimensional ZK Eq. (3.22) (only shows the shape of Eq. (3.38) with wave speed $\lambda = 3$, $a = 1$, $y = 1$, $z = 2$ and $-10 \leqslant x$, $t \leqslant 10$) (see Fig. 7).

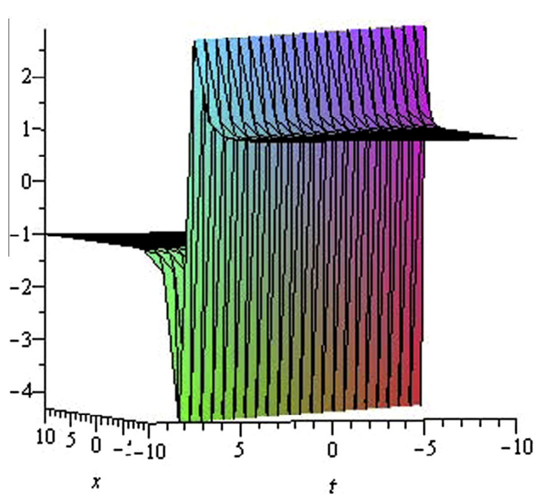

Figure 1 The shape of Eq. (3.18) with y = 2, $\lambda = -1$, $\alpha = 1$, $\beta = -1$ and $-10 \leqslant x,t \leqslant 10$.

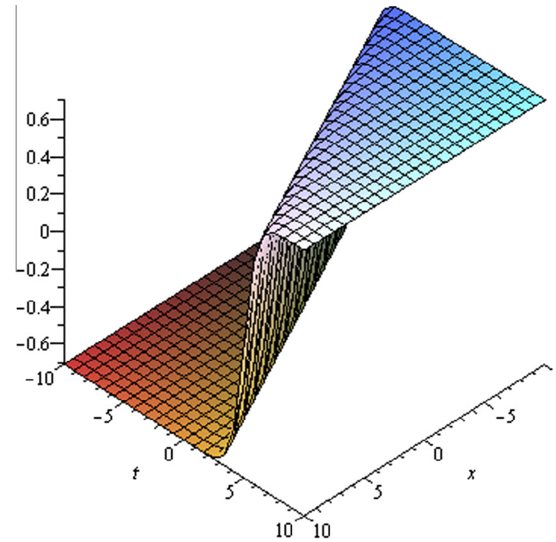

Figure 2 The shape of Eq. (3.19) with $y = 0, \lambda = 2, \alpha = -1, \beta = 2$ and $-10 \leqslant x,t \leqslant 10$.

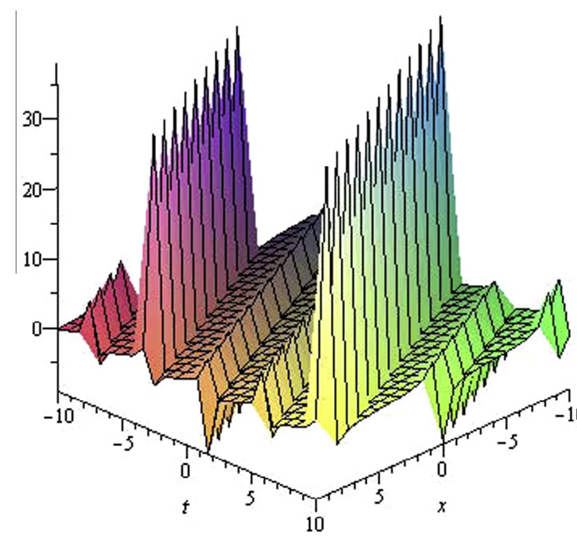

Figure 3 The shape of Eq. (3.20) with $\lambda = 2$, $y = 2$, $\alpha = 0.5$, $\beta = 1$ and $-10 \leqslant x,t \leqslant 10$.

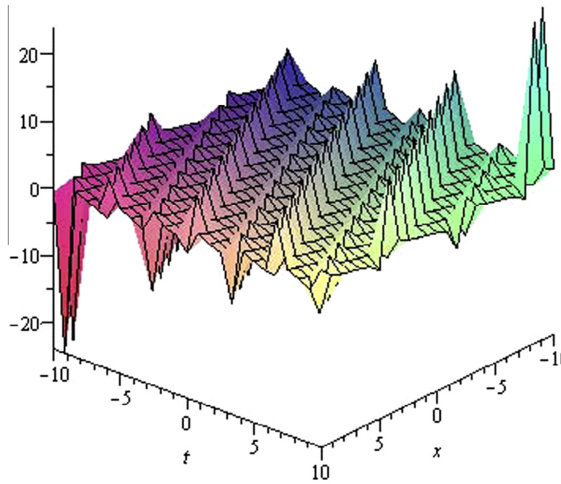

Figure 4 The shape of Eq. (3.21) with $\lambda = 2$, $y = 0$, $\alpha = 1$, $\beta = 2$ and $-10 \leqslant x,t \leqslant 10$.

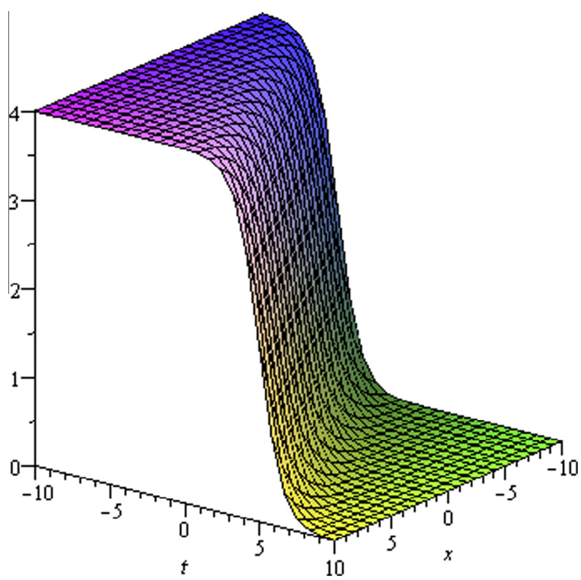

Figure 5 The shape of Eq. (3.37) with $\lambda = 2$, $a = 1$, $y = z = 0$, $-10 \leqslant x,t \leqslant 10$.

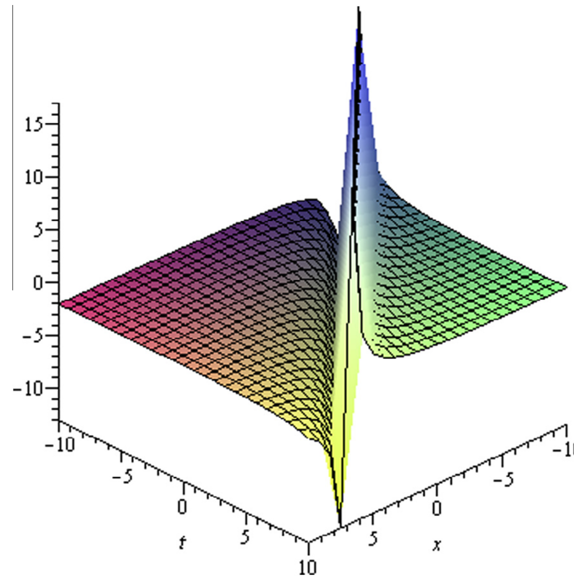

Figure 6 The shape of Eq. (3.38) with $\lambda = 1$, $a = -1$, $y = 1$, $z = 2$ and $-10 \leqslant x,t \leqslant 10$.

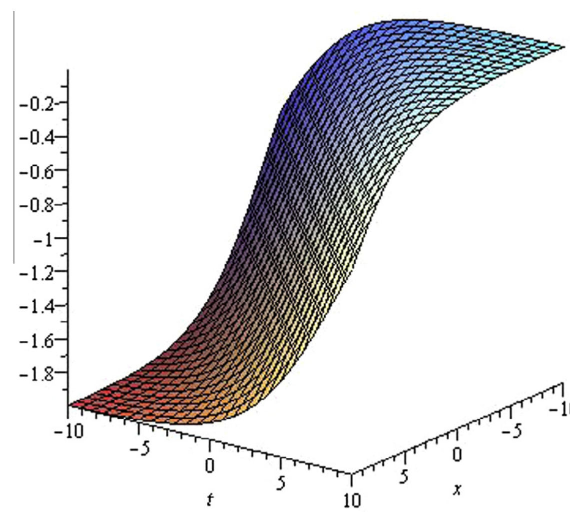

Figure 7 The shape of Eq. (3.39) with $\lambda = 1$, $a = -1$, $a_1 = 1$, $a_2 = -1$, $y = z = 1$, $-10 \leqslant x,t \leqslant 10$.

5. Comparison

Zayed (2011b) studied solutions of the (2 + 1)-dimensional cKG equation using the (G'/G)-expansion method combined with the Riccati equation. The solutions of the (2 + 1)-dimensional cKG equation obtained by MSE method are different from those of the (G'/G)-expansion method combined with the Riccati equation (see Appendix). Comparing the MSE method with the (G'/G)-expansion method we might conclude that the exact solutions have investigated using the (G'/G)-expansion method with the help of the symbolic computation software, such as, Mathematica, Maple, to facilitate the complex algebraic computations. On the other hand via the MSE method the exact solutions to these equations have been achieved without using any symbolic computation software because the method is very simple and easy for computations. Moreover, in Exp-function method, tanh function method, Jacobi elliptic function method, (G'/G)-expansion method, etc., $\phi(\xi)$ is a pre-defined function or solution of a pre-defined differential equation. But, since in the MSE method $\phi(\xi)$ is not pre-defined or not solution of any pre-defined equation, some fresh solutions might be found. Consequently, much new and more general exact solutions of the (2 + 1)-dimensional cKG equation can be obtained by means of the MSE method with less effort.

6. Conclusions

In this article, we have found the traveling wave solutions of the (2 + 1)-dimensional cubic Klein–Gordon (cKG) equation and (3+1)-dimensional Zakharov–Kuznetsov (ZK) equation using the MSE method. These traveling wave solutions are expressed in terms of hyperbolic, and trigonometric functions involving arbitrary parameters. When these parameters are taken special values, the solitary waves are originated from the traveling waves. Compared to the methods used before, one can see that this method is direct, concise and effective. This method can also be used to many other nonlinear equations.

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Appendix A. Zayed (2011b) studied solutions of the (2+1)-dimensional cKG equation using the (G'/G)-expansion method and achieved the following calculations and exact solutions:

$$\begin{aligned}&G^{-1}\left[2\alpha_1A^2B(2-V^2)+\alpha\alpha_1A+3\beta\alpha_1^3A^2B+3\beta\alpha_0^2\alpha_1A\right]\\&+G\left[2\alpha_1AB^2(2-V^2)+\alpha\alpha_1B+3B\beta\alpha_0^2\alpha_1+3\beta\alpha_1^3AB^2\right]\\&+G^{-2}\left[3A^2\beta\alpha_1^2\alpha_0\right]+G^2[3B^2\beta\alpha_1^2\alpha_0]\\&+G^3\left[\beta\alpha_1^3B^3+2\alpha_1B^3(2-V^2)\right]+G^{-3}\left[\beta\alpha_1^3A^3+2\alpha_1A^3(2-V^2)\right]\\&+\alpha\alpha_0+\beta\alpha_0^3=0. \qquad (34)\end{aligned}$$

Consequently the following algebraic equations

$$\begin{aligned}&2\alpha_1A^2B(2-V^2)+\alpha\alpha_1A+3\beta\alpha_1^3A^2B+3\beta\alpha_0^2\alpha_1A=0,\\&2\alpha_1AB^2(2-V^2)+\alpha\alpha_1B+3B\beta\alpha_0^2\alpha_1+3\beta\alpha_1^3AB^2=0,\\&3A^2\beta\alpha_1^2\alpha_0=0,\\&3B^2\beta\alpha_1^2\alpha_0=0,\\&\beta\alpha_1^3B^3+2\alpha_1B^3(2-V^2)=0,\\&\beta\alpha_1^3A^3+2\alpha_1A^3(2-V^2)=0,\\&\alpha\alpha_0+\beta\alpha_0^3=0. \qquad (35)\end{aligned}$$

Which can be solved to get

$$\alpha_1=\pm\sqrt{\frac{\alpha}{2\beta AB}},\quad \alpha_0=0,\quad V=\pm\sqrt{2-\frac{\alpha}{4AB}}. \qquad (36)$$

Substituting (3.36) into (3.4) yields

$$u(\xi)=\pm\sqrt{\frac{\alpha}{2\beta AB}}\left(\frac{G\prime}{G}\right), \qquad (37)$$

where

$$\xi=x+y\mp t\sqrt{2-\frac{\alpha}{4AB}}. \qquad (38)$$

According to general solutions of the Riccati equations, he (Zayed, 2011b) got the following families of exact solutions:

Family 1. If $A=\frac{1}{2}, B=-\frac{1}{2}$, then

$$u(\xi)=-\sqrt{\frac{2\alpha}{\beta}}\,\mathrm{isech}\xi, \qquad (39)$$

Or

$$u(\xi)=\pm\sqrt{\frac{2\alpha}{\beta}}\,\mathrm{csch}\xi, \qquad (40)$$

where $\xi=x+y\mp t\sqrt{2+\alpha}$.

Family 2. If $A=B=\pm\frac{i}{2}$, then

$$u(\xi)=\sqrt{\frac{-2\alpha}{\beta}}\sec\xi, \qquad (41)$$

Or

$$u(\xi)=\pm\sqrt{\frac{-2\alpha}{\beta}}\,\csc\xi, \qquad (42)$$

where $\xi=x+y\mp t\sqrt{2+\alpha}$.

Family 3. If $A=1, B=-1$, then

$$u(\xi)=\pm\sqrt{\frac{\alpha}{2\beta}}(\coth\xi-\tanh\xi), \qquad (43)$$

where $\xi=x+y\mp t\sqrt{2+\frac{\alpha}{4}}$.

Family 4. If $A=B=1$, then

$$u(\xi)=\pm\sqrt{\frac{-\alpha}{2\beta}}(\cot\xi+\tan\xi), \qquad (44)$$

where $\xi=x+y\mp t\sqrt{2-\frac{\alpha}{4}}$.

The general solutions of the Riccati equations $G'(\xi)=A+BG^2$ are well known which are listed in the following table:

A	B	The solution $G(\xi)$
$\frac{1}{2}$	$-\frac{1}{2}$	$tanh\xi\pm isech\xi,\quad coth\xi\pm csch\xi,\quad tanh\frac{\xi}{2}, coth\frac{\xi}{2}$
$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	$sec\xi\pm tan\xi,\quad \pm tan\frac{\xi}{2},\quad \mp cot\frac{\xi}{2}, \pm(csc\xi-cot\xi)$
1	−1	$tanh\xi,\quad coth\xi$
1	1	$tan\xi,\quad -cot\xi$
0	≠0	$-\frac{1}{B\xi+c_1}$, where c_1 is an arbitrary constant.

References

Abdou, M.A., 2007. The extended tanh-method and its applications for solving nonlinear physical models. App. Math. Comput. 190, 988–996.

Adomian, G., 1994. Solving Frontier Problems of Physics: The Decomposition Method. Kluwer Academic, Boston, MA.

Akbar, M.A., Ali, N.H.M., 2011a. The alternative (G'/G)-expansion method and its applications to nonlinear partial differential equations. Int. J. Phys. Sci. 6 (35), 7910–7920.

Akbar, M.A., Ali, N.H.M., 2011b. Exp-function method for Duffing equation and new solutions of (2+1) dimensional dispersive long wave equations. Prog. Appl. Math. 1 (2), 30–42.

Akbar, M.A., Ali, N.H.M., Zayed, E.M.E., 2012a. Abundant exact traveling wave solutions of the generalized Bretherton equation via (G'/G)-expansion method. Commun. Theor. Phys. 57, 173–178.

Akbar, M.A., Ali, N.H.M., Zayed, E.M.E., 2012b. A generalized and improved (G′/G)-expansion method for nonlinear evolution equations. Math. Prob. Engr. 2012, 22. http://dx.doi.org/10.1155/2012/459879.

Akbar, M.A., Ali, N.H.M., Mohyud-Din, S.T., 2012c. The alternative (G'/G)-expansion method with generalized Riccati equation: application to fifth order (1+1)-dimensional Caudrey–Dodd–Gibbon equation. Int. J. Phys. Sci. 7 (5), 743–752.

Akbar, M.A., Ali, N.H.M., Mohyud-Din, S.T., 2012d. Some new exact traveling wave solutions to the (3+1)-dimensional Kadomtsev–Petviashvili equation. World Appl. Sci. J. 16 (11), 1551–1558.

Ali, A.T., 2011. New generalized Jacobi elliptic function rational expansion method. J. Comput. Appl. Math. 235, 4117–4127.

Fan, E.G., 2000. Extended tanh-method and its applications to nonlinear equations. Phys. Lett. A 277, 212–218.
Gepreel, K.A., 2011a. Exact complexiton soliton solutions for nonlinear partial differential equations. Int. Math. Forum 6 (26), 1261–1272.
Gepreel, K.A., 2011b. A generalized (G′/G)-expansion method for finding traveling wave solutions of nonlinear evolution equations. J. Partial Diff. Eq. 24, 55–69.
Gepreel, K.A., 2011c. Exact solutions for nonlinear PDEs with the variable coefficients in mathematical physics. J. Inf. Comput. Sci. 6, 003–014.
Gepreel, K.A., Shehata, A.R., 2012. Exact complexiton soliton solutions for nonlinear partial differential equations in mathematical physics. Sci. Res. Essays 7 (2), 149–157.
He, J.H., Wu, X.H., 2006. Exp-function method for nonlinear wave equations. Chaos Solitons Fract. 30, 700–708.
He, Y., Li, S., Long, Y., 2012. Exact solutions of the Klein–Gordon equation by modified Exp-function method. Int. Math. Forum 7 (4), 175–182.
Hirota, R., 1973. Exact envelope soliton solutions of a nonlinear wave equation. J. Math. Phys. 14, 805–810.
Hirota, R., Satsuma, J., 1981. Soliton solutions of a coupled KDV equation. Phys. Lett. A 85, 404–408.
Jawad, A.J.M., Petkovic, M.D., Biswas, A., 2010. Modified simple equation method for nonlinear evolution equations. Appl. Math. Comput. 217, 869–877.
Liang, M.S. et al, 2011. A method to construct Weierstrass elliptic function solution for nonlinear equations. Int. J. Modern Phys. B 25 (4), 1931–1939.
Malfliet, M., 1992. Solitary wave solutions of nonlinear wave equations. Am. J. Phys. 60, 650–654.
Mohiud-Din, S.T., 2007. Homotopy perturbation method for solving fourth-order boundary value problems. Math. Prob. Engr. 2007, 1–15. http://dx.doi.org/10.1155/2007/98602. Article ID 98602.
Naher, H., Abdullah, A.F., Akbar, M.A., 2011. The Exp-function method for new exact solutions of the nonlinear partial differential equations. Int. J. Phys. Sci. 6 (29), 6706–6716.
Naher, H., Abdullah, A.F., Akbar, M.A., 2012. New traveling wave solutions of the higher dimensional nonlinear partial differential equation by the Exp-function method. J. Appl. Math., 14. http://dx.doi.org/10.1155/2012/575387. Article ID 575387.
Nassar, H.A., Abdel-Razek, M.A., Seddeek, A.K., 2011. Expanding the tanh-function method for solving nonlinear equations. Appl. Math. 2, 1096–1104.
Shehata, A.R., 2010. The traveling wave solutions of the perturbed nonlinear Schrodinger equation and the cubic–quintic Ginzburg Landau equation using the modified (G′/G)-expansion method. Appl. Math. Comput. 217, 1–10.
Sirendaoreji, 2004. New exact travelling wave solutions for the Kawahara and modified Kawahara equations. Chaos Solitons Fract. 19, 147–150.
Wang, M., 1995. Solitary wave solutions for variant Boussinesq equations. Phys. Lett. A 199, 169–172.
Wang, M., Li, X., Zhang, J., 2008. The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. Phys. Lett. A 372, 417–423.
Zayed, E.M.E., 2010. Traveling wave solutions for higher dimensional nonlinear evolution equations using the (G′/G)-expansion method. J. Appl. Math. Informatics 28, 383–395.
Zayed, E.M.E., 2011a. A note on the modified simple equation method applied to Sharma–Tasso–Olver equation. Appl. Math. Comput. 218, 3962–3964.
Zayed, E.M.E., 2011b. The (G'/G)-expansion method combined with the Riccati equation for finding exact solutions of nonlinear PDEs. J. Appl. Math. Informatics 29, 351–367.
Zayed, E.M.E., Arnous, A.H., 2012. Exact solutions of the nonlinear ZK–MEW and the Potential YTSF equations using the modified simple equation method. AIP Conf. Proc. 1479, 2044–2048.
Zayed, E.M.E., Gepreel, K.A., 2009. The (G′/G)-expansion method for finding the traveling wave solutions of nonlinear partial differential equations in mathematical physics. J. Math. Phys. 50, 013502–013514.
Zayed, E.M.E., Gepreel, K.A., 2011. A series of complexiton soliton solutions for nonlinear Jaulent–Miodek PDEs using Riccati equations method. Proc. R. Soc. Edinburgh A Math. 141A, 1001–1015.
Zayed, E.M.E., Ibrahim, S.A.H., 2012. Exact solutions of nonlinear evolution equations in mathematical physics using the modified simple equation method. Chin. Phys. Lett. 29 (6), 060201.
Zayed, E.M.E., Zedan, H.A., Gepreel, K.A., 2004. On the solitary wave solutions for nonlinear Hirota–Sasuma coupled KDV equations. Chaos Solitons Fract. 22, 285–303.
Zhou, Y.B., Wang, M.L., Wang, Y.M., 2003. Periodic wave solutions to coupled KdV equations with variable coefficients. Phys. Lett. A 308, 31–36.