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لونية لعائلة من الرسومات البيانية ذات الخمس تقسيمات

R. Hasni[1], A. Shaman[2], G.C. Lau[3]

[1] Department of Mathematics, Faculty of Science and Technology, University Malaysia Terengganu, 21030 Kuala Terengganu, Terengganu, Malaysia
[2] Department of Physics, College of Science, Anbar University, Anbar, Iraq
[3] Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA (Segamat Campus), Johor, Malaysia

الملخص:

تم تحديد دالة P(G,λ) لتكون متعددة حدود لونية لرسم بياني نوع G. كما تم اعتبار رسميين بيانيين نوع Gو H بأنهما متعادلة لونيا بحيث P(G,λ)=P(H,λ). في هذا البحث تم تحديد رسومات بيانية معينة ذات خمس تقسيمات كاملة للرسم البياني G وذات خمس رؤوس وفقا للعدد 6 لتقسيمات مستقلة من الرسم البياني G. باستخدام هذه النتائج، تم فحص اللونية للرسم البياني نوع G مع نجوم معينة أو أجزاء مناظرة محذوفة، وكنتيجة، تم الحصول على عائلتين جديدتين وحيدية اللونية وكاملة التقسيمات الخمس للرسم البياني G وذات نجوم معينة او أجزاء مناظرة محذوفة.

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ORIGINAL ARTICLE

Chromaticity of a family of 5-partite graphs

R. Hasni [a,*], A. Shaman [b], G.C. Lau [c]

[a] Department of Mathematics, Faculty of Science and Technology, University Malaysia Terengganu, 21030 Kuala Terengganu, Terengganu, Malaysia
[b] Department of Physics, College of Science, Anbar University, Anbar, Iraq
[c] Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA (Segamat Campus), Johor, Malaysia

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Abstract Let $P(G,\lambda)$ be the chromatic polynomial of a graph G. Two graphs G and H are said to be chromatically equivalent, denoted $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. We write $[G] = \{H| H \sim G\}$. If $[G] = \{G\}$, then G is said to be chromatically unique. In this paper, we first characterize certain complete 5-partite graphs G with $5n$ vertices according to the number of 6-independent partitions of G. Using these results, we investigate the chromaticity of G with certain stars or matching deleted parts . As a by-product, two new families of chromatically unique complete 5-partite graphs G with certain stars or matching deleted parts are obtained.
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1. Introduction

All graphs considered here are simple and finite. For a graph G, let $P(G,\lambda)$ be the chromatic polynomial of G. Two graphs G and H are said to be *chromatically equivalent* (or simply χ-equivalent), symbolically $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply χ-unique) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be χ-closed. Many families of χ-unique graphs are known (Koh and Teo, 1990, 1997).

For a graph G, let $V(G)$, $E(G)$, $t(G)$ and $\chi(G)$ be the vertex set, edge set, number of triangles and chromatic number of G, respectively. Let O_n be an edgeless graph with n vertices. Let $Q(G)$ and $K(G)$ be the number of induced subgraphs isomorphic to C_4 and complete subgraphs K_4 in G. Let S be a set of s edges in G. By $G - S$ (or $G - s$) we denote the graph obtained from G by deleting all edges in S, and $\langle S\rangle$ the graph induced by S. For $t \geqslant 2$ and $1 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_t$, let $K(n_1,n_2,\ldots,n_t)$ be a complete t-partite graph with partition sets V_i such that $|V_i| = n_i$ for $i = 1, 2,\ldots,t$. In (Dong et al., 2001; Lau and Peng, 2010a,b, Lau et al., 2010; Roslan et al., 2010, 2011a, 2012a; Zhao et al., 2004; Zhao, 2004, 2005), the authors proved that certain families of complete t-partite graphs ($t = 2, 3, 4, 5$) with a matching or a star deleted are χ-unique. The case for the complete 6-partite graphs has been investigated in (2011c; 2012b; 2012c). In particular, Zhao et al. (2004) and Zhao (2005) investigated the chromaticity of complete 5-partite graphs G of $5n$ and $5n + 4$ vertices with certain stars or matching deleted parts. Roslan et al. (2011b) studied the chromaticity of complete 5-partite graphs G with $5n + i$

* Corresponding author. Tel.: +609 6683857.
E-mail address: hroslan@umt.edu.my (R. Hasni).
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vertices for $i = 1, 2, 3$ with certain stars or matching deleted parts. As a continuation, in this paper, we characterize certain complete 5-partite graphs G with $5n$ vertices according to the number of 6-independent partitions of G. Using these results, we investigate the chromaticity of G with certain stars or matching deleted parts. As a by-product, two new families of chromatically unique complete 5-partite graphs with certain stars or matching deleted parts are obtained. These results generalized Theorems 3 and 4 in (Zhao, 2005).

2. Some lemmas and notations

Let $\mathscr{K}^{-s}(n_1, n_2, \ldots, n_t)$ be the family $\{K(n_1,n_2,\ldots,n_t) - S | S \subset E(K(n_1,n_2,\ldots,n_t))$ and $|S| = s\}$. For $n_1 \geqslant s + 1$, we denote by $K_{i,j}^{-K_{1,s}}(n_1, n_2, \ldots, n_t)$ (respectively, $K_{i,j}^{-sK_2}(n_1, n_2, \ldots, n_t)$) the graph in $K^{-s}(n_1,n_2,\ldots,n_t)$ where the s edges in S induce a $K_{1,s}$ with center in V_i and all the end vertices in V_j (respectively, a matching with end vertices in V_i and V_j).

For a graph G and a positive integer r, a partition $\{A_1,A_2,\ldots,A_r\}$ of $V(G)$, where r is a positive integer, is called an *r-independent partition* of G if every A_i is independent of G. Let $\alpha(G,r)$ denote the number of r-independent partitions in G. Then, we have $P(G,\lambda) = \sum_{r=1}^{p} \alpha(G,r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-r+1)$ and p is the number of vertices of G(see Read and Tutte, 1988). Therefore, $\alpha(G,r) = \alpha(H,r)$ for each $r = 1, 2,\ldots$, if $G \sim H$.

For a graph G with p vertices, the polynomial $\sigma(G,x) = \sum_{r=1}^{p} \alpha(G,r)x^r$ is called the σ-polynomial of G (see Brenti, 1992). Clearly, $P(G,\lambda) = P(H,\lambda)$ implies that $\sigma(G,x) = \sigma(H,x)$ for any graphs G and H.

For disjoint graphs G and H, $G + H$ denotes the disjoint union of G and H. The join of G and H denoted by $G \vee H$ is defined as follows: $V(G \vee H) = V(G) \cup V(H)$; $E(G \vee H) = E(G) \cup E(H) \cup \{xy | x \in V(G), y \in V(H)\}$. For notations and terminology not defined here, we refer to (West, 2001).

Lemma 2.1. (Koh and Teo, 1990) *Let G and H be two graphs with $H \sim G$, then $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$, $t(G) = t(H)$ and $\chi(G) = \chi(H)$. Moreover, $\alpha(G,r) = \alpha(H,r)$ for $r \geqslant 1$, and $2K(G) - Q(G) = 2K(H) - Q(H)$. Note that if $\chi(G) = 3$, then $G \sim H$ implies that $Q(G) = Q(H)$.*

Lemma 2.2. (Brenti, 1992) *Let G and H be two disjoint graphs. Then*

$$\sigma(G \vee H, x) = \sigma(G,x)\sigma(H,x).$$

In particular,

$$\sigma(K(n_1, n_2, \ldots, n_t), x) = \prod_{i=1}^{t} \sigma(O_{n_i}, x).$$

Lemma 2.3 Zhao et al., 2004. *Let $G = K(n_1,n_2,n_3,n_4,n_5)$ and S be a set of some s edges of G. If $H \sim G - S$, then there is a complete graph $F = K(p_1,p_2,p_3,p_4,p_5)$ and a subset S' of $E(F)$ of some s' edges of F such that $H = F - S'$ with $|S'| = s' = e(F) - e(G) + s$.*

Let $n_1 \leqslant n_2 \leqslant n_3 \leqslant n_4 \leqslant n_5$ be positive integers and $\{n_{i_1}, n_{i_2}, n_{i_3}, n_{i_4}, n_{i_5}\} = \{n_1, n_2, n_3, n_4, n_5\}$. If there exist two elements n_{i_1} and n_{i_2} in $\{n_1,n_2,n_3,n_4,n_5\}$ such that $n_{i_2} - n_{i_1} \geqslant 2$, $H' = K(n_{i_1} + 1, n_{i_2} - 1, n_{i_3}, n_{i_4}, n_{i_5})$ is called an *improvement* of $H = K(n_1,n_2,n_3,n_4,n_5)$.

Lemma 2.4 Zhao et al., 2004. *Suppose $n_1 \leqslant n_2 \leqslant n_3 \leqslant n_4 \leqslant n_5$ and $H' = K(n_{i_1} + 1, n_{i_2} - 1, n_{i_3}, n_{i_4}, n_{i_5}\}$ is an improvement of $H = K(n_1,n_2,n_3,n_4,n_5)$, then*

$$\alpha(H,6) - \alpha(H',6) = 2^{n_{i_2}-2} - 2^{n_{i_1}-1} \geqslant 2^{n_{i_1}-1}.$$

Let $G = K(n_1,n_2,n_3,n_4,n_5)$. For a graph $H = G - S$, where S is a set of some s edges of G, define $\alpha'(H) = \alpha(H,6) - \alpha(G,6)$. Clearly, $\alpha'(H) \geqslant 0$.

Lemma 2.5 Zhao et al., 2004. *Let $G = K(n_1,n_2,n_3,n_4,n_5)$. Suppose that $\min \{n_i | i = 1, 2, 3, 4, 5\} \geqslant s + 1 \geqslant 1$ and $H = G - S$, where S is a set of some s edges of G, then*

$$s \leqslant \alpha'(H) = \alpha(H,6) - \alpha(G,6) \leqslant 2^s - 1,$$

$\alpha'(H) = s$ if and only if the set of end-vertices of any $r \geqslant 2$ edges in S is not independent in H, and $\alpha'(H) = 2^s - 1$ if and only if S induces a star $K_{1,s}$ and all vertices of $K_{1,s}$ other than its center belong to a same A_i.

Lemma 2.6 Dong et al., 2001. *Let n_1,n_2 and s be positive integers with $3 \leqslant n_1 \leqslant n_2$, then*

(1) $K_{1,2}^{-K_{1,s}}(n_1, n_2)$ is χ-unique for $1 \leqslant s \leqslant n_2 - 2$,
(2) $K_{2,1}^{-K_{1,s}}(n_1, n_2)$ is χ-unique for $1 \leqslant s \leqslant n_1 - 2$, and
(3) $K^{-sK_2}(n_1, n_2)$ is χ-unique for $1 \leqslant s \leqslant n_1 - 1$.

For a graph $G \in K^{-s}(n_1,n_2,\ldots,n_t)$, we say an induced C_4 subgraph of G is of Type 1 (respectively Type 2 and Type 3) if the vertices of the induced C_4 are in exactly two (respectively three and four) partite sets of $V(G)$. An example of induced C_4 of Types 1, 2 and 3 is shown in Fig. 1.

Suppose G is a graph in $K^{-s}(n_1,n_2,\ldots,n_t)$. Let $S_{ij}(1 \leqslant i \leqslant t, 1 \leqslant j \leqslant t)$ be a subset of S such that each edge in S_{ij} has an end-vertex in V_i and another end-vertex in V_j with $|S_{ij}| = s_{ij} \geqslant 0$.

Lemma 2.7 Lau and Peng, 2010b. *For integer $t \geqslant 3$, let $F = K(n_1,n_2,\ldots,n_t)$ be a complete t-partite graph and let $G = F - S$, where S is a set of s edges in F. If S induces a matching in F, then*

$$Q(G) = Q(F) - \sum_{1 \leqslant i<j \leqslant t} (n_i - 1)(n_j - 1)s_{ij} + \binom{s}{2} - \sum_{1 \leqslant i<j<l \leqslant t} s_{ij}s_{il} - \sum_{\substack{1 \leqslant i<j \leqslant t \\ 1 \leqslant k<l \leqslant t \\ i<k}} s_{ij}s_{kl} + \sum_{1 \leqslant i<j \leqslant t} \left[s_{ij} \sum_{k \notin \{i,j\}} \binom{n_k}{2} \right] + \sum_{\substack{1 \leqslant i<j \leqslant t \\ 1 \leqslant i<k<l \leqslant t \\ j \notin \{k,l\}}} s_{ij}s_{kl},$$

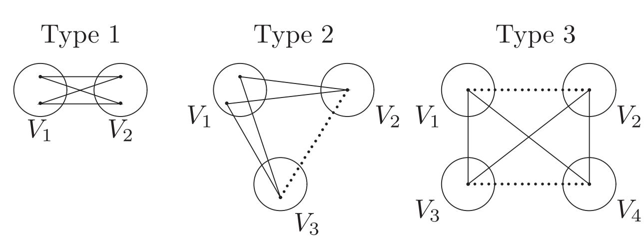

Figure 1 Three types of induced C_4.

and

$$K(G) = K(F) - \sum_{1\leqslant i<j\leqslant t}\left[s_{ij}\sum_{\substack{1\leqslant k<l\leqslant t\\ \{i,j\}\cap\{k,l\}=\emptyset}} n_k n_l\right] + \sum_{\substack{1\leqslant i<j\leqslant t\\ 1\leqslant i<k<l\leqslant t\\ j\notin\{k,l\}}} s_{ij}s_{kl}.$$

By using Lemma 2.7, we obtain the following.

Lemma 2.8. *Let $F = K(n_1,n_2,n_3,n_4,n_5)$ be a complete 5-partite graph and let $G = F - S$ where S is a set of s edges in F. If S induces a matching in F, then*

$$\begin{aligned}Q(G) = Q(F) &- \sum_{1\leqslant i<j\leqslant 5}(n_i-1)(n_j-1)s_{ij} + \binom{s}{2} - s_{12}(s_{13}+s_{14}\\ &+ s_{15}+s_{23}+s_{24}+s_{25}) - s_{13}(s_{14}+s_{15}+s_{23}\\ &+ s_{34}+s_{35}) - s_{14}(s_{15}+s_{24}+s_{34}+s_{45}) - s_{15}(s_{25}\\ &+ s_{35}+s_{45}) - s_{23}(s_{24}+s_{25}+s_{34}+s_{35}) - s_{24}(s_{25}+s_{34}\\ &+ s_{45}) - s_{25}(s_{35}+s_{45}) - s_{34}(s_{35}+s_{45}) - s_{35}s_{45}\\ &+ \sum_{1\leqslant i<j\leqslant 5}\left[s_{ij}\sum_{k\notin\{i,j\}}\binom{n_k}{2}\right],\end{aligned}$$

and

$$\begin{aligned}K(G) = K(F) &- \sum_{1\leqslant i<j\leqslant 5}\left[s_{ij}\sum_{\substack{1\leqslant k<l\leqslant 5\\ \{i,j\}\cap\{k,l\}=\emptyset}} n_k n_l\right] + s_{12}(s_{34}+s_{35}\\ &+ s_{45}) + s_{13}(s_{24}+s_{25}+s_{45}) + s_{14}(s_{23}+s_{25}+s_{35})\\ &+ s_{15}(s_{23}+s_{24}+s_{34}) + s_{23}s_{45} + s_{24}s_{35} + s_{25}s_{34}.\end{aligned}$$

3. Characterization

In this section, we shall characterize certain complete 5-partite graphs $G = K(n_1,n_2,n_3,n_4,n_5)$ according to the number of 6-independent partitions of G where $n_5 - n_1 \leqslant 4$.

Theorem 3.1. *Let $G = K(n_1,n_2,n_3,n_4,n_5)$ be a complete 5-partite graph such that $n_1 + n_2 + n_3 + n_4 + n_5 = 5n$ and $n_5 - n_1 \leqslant 4$. Define $\theta(G) = [\alpha(G,6) - 2^{n+1} - 2^{n-1} + 5]/2^{n-2}$. Then*

(i) $\theta(G) = 0$ if and only if $G = K(n,n,n,n,n)$;
(ii) $\theta(G) = 1$ if and only if $G = K(n-1,n,n,n,n+1)$;
(iii) $\theta(G) = 2$ if and only if $G = K(n-1,n-1,n,n+1,n+1)$;
(iv) $\theta(G) = 2\frac{1}{2}$ if and only if $G = K(n-2,n,n,n+1,n+1)$;
(v) $\theta(G) = 3\frac{1}{2}$ if and only if $G = K(n-2,n-1,n+1,n+1,n+1)$;
(vi) $\theta(G) = 4$ if and only if $G = K(n-1,n-1,n,n,n+2)$;
(vii) $\theta(G) = 4\frac{1}{4}$ if and only if $G = K(n-3,n,n+1,n+1,n+1)$;
(viii) $\theta(G) = 4\frac{1}{2}$ if and only if $G = K(n-2,n,n,n,n+2)$;
(ix) $\theta(G) = 5$ if and only if $G = K(n-1,n-1,n-1,n+1,n+2)$;
(x) $\theta(G) = 5\frac{1}{2}$ if and only if $G = K(n-2,n-1,n,n+1,n+2)$;
(xi) $\theta(G) = 7$ if and only if $G = K(n-2,n-2,n+1,n+1,n+2)$;
(xii) $\theta(G) = 8\frac{1}{2}$ if and only if $G = K(n-2,n-1,n-1,n+2,n+2)$;
(xiii) $\theta(G) = 9$ if and only if $G = K(n-2,n-2,n,n+2,n+2)$;
(xiv) $\theta(G) = 11$ if and only if $G = K(n-1,n-1,n-1,n,n+3)$;

Proof. In order to complete the proof of the theorem, we first give a table for the θ-value of various complete 5-partite graphs with $5n$ vertices as shown in Table 1.

By using Table 1, Lemma 2.4 and the definition of improvement, the proof is complete. □

4. Chromatically closed 5-partite graphs

In this section, we obtain a χ-closed family of graphs from the graphs in Theorem 3.1.

Theorem 4.1. *The family of graphs $\mathscr{K}^{-s}(n_1,n_2,n_3,n_4,n_5)$ where $n_1 + n_2 + n_3 + n_4 + n_5 = 5n$, $n_5 - n_1 \leqslant 4$ and $n_1 \geqslant s + 5$ is χ-closed.*

Proof. By Theorem 3.1, there are 14 cases to consider. Denote each graph in Theorem 3.1 $(i),(ii),\ldots,(xiv)$ by $G_1,G_2,\ldots,G_{14}$, respectively. Suppose $H \sim G_i - S$. It suffices to show that $H \in \{G_i - S\}$. By Lemma 2.3, we know there exists a complete 5-partite graph $F = (p_1,p_2,p_3,p_4,p_5)$ such that $H = F - S'$ with $|S'| = s' = e(F) - e(G) + s \geqslant 0$.

Case (i). Let $G = G_1$ with $n \geqslant s + 2$. In this case, $H \sim F - S \in \mathscr{K}^{-s}(n,n,n,n,n)$. By Lemma 2.5, we have

$$\alpha(G-S,6) = \alpha(G,6) + \alpha'(G-S) \text{ with } s \leqslant \alpha'(G-S) \leqslant 2^s - 1,$$
$$\alpha(F-S',6) = \alpha(F,6) + \alpha'(F-S') \text{ with } 0 \leqslant s' \leqslant \alpha'(F-S').$$

Hence,

$$\alpha(F-S',6) - \alpha(G-S,6) = \alpha(F,6) - \alpha(G,6) + \alpha'(F-S') - \alpha'(G-S).$$

By the definition, $\alpha(F,6) - \alpha(G,6) = 2^{n-2}(\theta(F) - \theta(G))$. By Theorem 3.1, $\theta(F) \geqslant 0$. Suppose $\theta(F) > 0$, then

$$\begin{aligned}\alpha(F-S',6) - \alpha(G-S,6) &\geqslant 2^{n-2} + \alpha'(F-S') - \alpha'(G-S)\\ &\geqslant 2^s + \alpha'(F-S') - 2^s + 1,\\ &\geqslant 1,\end{aligned}$$

contradicting $\alpha(F-S',6) = \alpha(G-S,6)$. Hence, $\theta(F) = 0$ and so $F = G$ and $s = s'$. Therefore, $H \in \mathscr{K}^{-s}(n,n,n,n,n)$.

Case (ii). Let $G = G_2$ with $n \geqslant s + 3$. In this case, $H \sim F - S \in \mathscr{K}^{-s}(n-1,n,n,n,n+1)$. By Lemma 2.5, we have

Table 1 Some complete 5-partite graphs with $5n$ vertices and their θ-values.

$G_i (1 \leqslant i \leqslant 23)$	$\theta(G_i)$	$G_i (24 \leqslant i \leqslant 46)$	$\theta(G_i)$
$G_1 = K(n,n,n,n,n)$	0	$G_{24} = K(n-4,n,n+1,n+1,n+2)$	$8\frac{1}{8}$
$G_2 = K(n-1,n,n,n,n+1)$	1	$G_{25} = K(n-4,n,n,n+2,n+2)$	$10\frac{1}{8}$
$G_3 = K(n-1,n-1,n,n+1,n+1)$	2	$G_{26} = K(n-4,n,n,n+1,n+3)$	$14\frac{1}{8}$
$G_4 = K(n-2,n,n,n+1,n+1)$	$2\frac{1}{2}$	$G_{27} = K(n-1,n-1,n-1,n-1,n+4)$	26
$G_5 = K(n-1,n-1,n,n,n+2)$	4	$G_{28} = K(n-2,n-1,n-1,n,n+4)$	$26\frac{1}{2}$
$G_6 = K(n-2,n,n,n,n+2)$	$4\frac{1}{2}$	$G_{29} = K(n-2,n-2,n-1,n+2,n+3)$	16
$G_7 = K(n-1,n-1,n-1,n+1,n+2)$	5	$G_{30} = K(n-3,n-1,n-1,n+2,n+3)$	$16\frac{1}{4}$
$G_8 = K(n-2,n-1,n+1,n+1,n+1)$	$3\frac{1}{2}$	$G_{31} = K(n-3,n-2,n+1,n+2,n+2)$	$10\frac{3}{4}$
$G_9 = K(n-2,n-1,n,n+1,n+2)$	$5\frac{1}{2}$	$G_{32} = K(n-3,n-2,n+1,n+1,n+3)$	$14\frac{3}{4}$
$G_{10} = K(n-3,n,n+1,n+1,n+1)$	$4\frac{1}{4}$	$G_{33} = K(n-4,n-1,n+1,n+2,n+2)$	$11\frac{1}{8}$
$G_{11} = K(n-3,n,n,n+1,n+2)$	$6\frac{1}{4}$	$G_{34} = K(n-4,n-1,n+1,n+1,n+3)$	$15\frac{1}{8}$
$G_{12} = K(n-1,n-1,n-1,n,n+3)$	11	$G_{35} = K(n-3,n-2,n,n+2,n+3)$	$16\frac{3}{4}$
$G_{13} = K(n-2,n-1,n,n,n+3)$	$11\frac{1}{2}$	$G_{36} = K(n-4,n-1,n,n+2,n+3)$	$17\frac{1}{8}$
$G_{14} = K(n-3,n,n,n,n+3)$	$12\frac{1}{4}$	$G_{37} = K(n-5,n+1,n+1,n+1,n+2)$	$10\frac{1}{16}$
$G_{15} = K(n-2,n-1,n-1,n+2,n+2)$	$8\frac{1}{2}$	$G_{38} = K(n-5,n,n+1,n+2,n+2)$	$12\frac{1}{16}$
$G_{16} = K(n-2,n-1,n-1,n+1,n+3)$	$12\frac{1}{2}$	$G_{39} = K(n-5,n,n+1,n+1,n+3)$	$16\frac{1}{16}$
$G_{17} = K(n-2,n-2,n+1,n+1,n+2)$	7	$G_{40} = K(n-5,n,n,n+2,n+3)$	$18\frac{1}{16}$
$G_{18} = K(n-3,n-1,n+1,n+1,n+2)$	$7\frac{1}{4}$	$G_{41} = K(n-3,n-3,n+2,n+2,n+2)$	$14\frac{1}{2}$
$G_{19} = K(n-2,n-2,n,n+2,n+2)$	9	$G_{42} = K(n-3,n-3,n+1,n+2,n+3)$	$18\frac{1}{2}$
$G_{20} = K(n-2,n-2,n,n+1,n+3)$	13	$G_{43} = K(n-4,n-2,n+2,n+2,n+2)$	$14\frac{5}{8}$
$G_{21} = K(n-3,n-1,n,n+2,n+2)$	$9\frac{1}{4}$	$G_{44} = K(n-4,n-2,n+1,n+2,n+3)$	$19\frac{1}{8}$
$G_{22} = K(n-3,n-1,n,n+1,n+3)$	$13\frac{1}{4}$	$G_{45} = K(n-6,n+1,n+1,n+2,n+2)$	$14\frac{1}{32}$
$G_{23} = K(n-4,n+1,n+1,n+1,n+1)$	$6\frac{1}{8}$	$G_{46} = K(n-6,n+1,n+1,n+1,n+3)$	$18\frac{1}{32}$

$$\alpha(G-S,6) = \alpha(G,6) + \alpha'(G-S) \text{ with } s \leqslant \alpha'(G-S) \leqslant 2^s - 1,$$
$$\alpha(F-S',6) = \alpha(F,6) + \alpha'(F-S') \text{ with } 0 \leqslant s' \leqslant \alpha'(F-S').$$

Hence,

$$\alpha(F-S',6) - \alpha(G-S,6) = \alpha(F,6) - \alpha(G,6) + \alpha'(F-S') - \alpha'(G-S).$$

By the definition, $\alpha(F,6) - \alpha(G,6) = 2^{n-2}(\theta(F) - \theta(G))$. Suppose $\theta(F) \neq \theta(G)$. Then, we consider two subcases.

Subcase (a). $\theta(F) < \theta(G)$. By Theorem 3.1, $F = G_1$ and $H = G_1 - S' \in \{G_1 - S'\}$. However, $G - S \notin \{G_1 - S'\}$ since by Case (i) above, $\{G_1 - S'\}$ is χ-closed, a contradiction.

Subcase (b). $\theta(F) > \theta(G)$. By Theorem 3.1, $\alpha(F,6) - \alpha(G,6) \geqslant 2^{n-2}$. So,

$$\alpha(F-S',6) - \alpha(G-S,6) \geqslant 2^{n-2} + \alpha'(F-S') - \alpha'(G-S) \geqslant 2^s + \alpha'(F-S') - 2^s + 1 \geqslant 1,$$

contradicting $\alpha(F-S',6) = \alpha(G-S,6)$. Hence, $\theta(F) - \theta(G) = 0$ and so $F = G$ and $s = s'$. Therefore, $H \in \mathscr{K}^{-s}(n-1,n,n,n,n+1)$.

Using Table 1, we can prove (iii) to (xiv) in a similar way. This completes the proof. □

5. Chromatically unique 5-partite graphs

The following results give two families of chromatically unique complete 5-partite graphs having $5n$ vertices with a set S of s edges deleted where the deleted edges induce a star $K_{1,s}$ and a matching sK_2, respectively.

Theorem 5.1. em The graphs $K_{i,j}^{-K_{1,s}}(n_1,n_2,n_3,n_4,n_5)$ where $n_1 + n_2 + n_3 + n_4 + n_5 = 5n$, $n_5 - n_1 \leqslant 4$ and $n_1 \geqslant s + 5$ are χ-unique for $1 \leqslant i \neq j \leqslant 5$.

Proof. By Theorem 3.1, there are 14 cases to consider. Denote each graph in Theorem 3.1 (*i*),(*ii*),…,(*xiv*) by $G_1,G_2,\ldots,G_{14}$, respectively. The proof for each graph obtained from G_i ($i = 1, 2,\ldots,14$) is similar, so we only give the detailed proof for the graphs obtained from G_2 below.

By Lemma 2.5 and Case 2 of Theorem 4.1, we know that $K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n+1) = \{K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n+1) | (i,j) \in \{(1,2),(2,1),(1,5),(5,1),(2,3),(4,5),(5,4)\}$ is χ-closed for $n \geqslant s+3$. Note that

$t(K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n+1)) = t(G_2) - s(3n+1)$ for $(i,j) \in \{(1,2),(2,1)\}$,
$t(K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n+1)) = t(G_2) - 3sn$ for $(i,j) \in \{(1,5),(5,1)\}$,
$t(K_{2,3}^{-K_{1,s}}(n-1,n,n,n,n+1)) = t(G_2) - 3sn$,
$t(K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n+1)) = t(G_2) - s(3n-1)$ for $(i,j) \in \{(4,5),(5,4)\}$.

By Lemmas 2.2 and 2.6, we conclude that $\sigma(K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n+1)) \neq \sigma(K_{j,i}^{-K_{1,s}}(n-1,n,n,n,n+1))$ for each $(i,j) \in \{(1,2),(1,5),(4,5)\}$. We now show that $K_{2,3}^{-K_{1,s}}(n-1,n,n,n,n+1)$ and $K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n+1)$ are not χ-equivalent for $(i,j) \in \{(1,5),(5,1)\}$. We have

$$Q(K_{2,3}^{-K_{1,s}}(n-1,n,n,n,n+1)) = Q(G_2) - s(n-1)^2 + \binom{s}{2} + s\left(\binom{n-1}{2} + \binom{n}{2} + \binom{n+1}{2}\right);$$

$$Q(K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n+1)) = Q(G_2) - sn(n-2) + \binom{s}{2} + 3s\binom{n}{2} \text{ for } (i,j) \in \{(1,5),(5,1)\};$$

with

$$Q(K_{2,3}^{-K_{1,s}}(n-1,n,n,n,n+1)) - Q(K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n+1)) = 0$$

since $s_{ij} = 0$ if $(i,j) \neq \{(2,3),(1,5),(5,1)\}$.

We also obtain

$$K(K_{2,3}^{-K_{1,s}}(n-1,n,n,n,n+1)) = K(G_2) - s(3n^2-1);$$
$$K(K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n+1)) = K(G_2) - 3sn^2 \text{ for } (i,j) \in \{(1,5),(5,1)\},$$

with

$$K(K_{2,3}^{-K_{1,s}}(n-1,n,n,n,n+1)) - K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n+1)) = s$$

since $s_{ij} = 0$ if $(i,j) \neq \{(2,3),(1,5),(5,1)\}$.

This means that $2K(K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n+1)) - Q(K_{i,j}^{-}K_{1,s}(n-1,n,n,n,n+1)) \neq 2K(K_{2,3}^{-K_{1,s}}(n-1,n,n,n,n+1)) - Q(K_{2,3}^{-K_{1,s}}(n-1,n,n,n,n+1))$, contradicting Lemma 2.1. Hence, $K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n+1)$ is χ-unique where $n \geqslant s+3$ for $1 \leqslant i \neq j \leqslant 5$.

The proof is thus complete. □

Theorem 5.2. *The graphs* $K_{1,2}^{-sK_2}(n_1,n_2,n_3,n_4,n_5)$ *where* $n_1+n_2+n_3+n_4+n_5 = 5n$, $n_5 - n_1 \leqslant 4$ *and* $n_1 \geqslant s+5$ *are* χ*-unique.*

Proof. By Theorem 3.1, there are 14 cases to consider. Denote each graph in Theorem 3.1 (*i*),(*ii*),…,(*xiv*) by $G_1, G_2, \ldots, G_{14}$, respectively. For a graph $K(p_1,p_2,p_3,p_4,p_5)$, let $S = \{e_1,e_2,\ldots,e_s\}$ be the set of s edges in $E(K(p_1,p_2,p_3,p_4,p_5))$ and let $t(e_i)$ denote the number of triangles containing e_i in $K(p_1,p_2,p_3,p_4,p_5)$. The proofs for each graph obtained from $G_i (i = 1, 2, \ldots, 14)$ are similar, so we only give the proof for the graph obtained from G_1 and G_2 as follows.

Suppose $H \sim G = K_{1,2}^{-sK_2}(n,n,n,n,n)$ for $n \geqslant s+2$. By Theorem 4.1 and Lemma 2.1, $H \in \mathscr{K}^{-s}(n,n,n,n,n)$ and $\alpha'(H) = \alpha'(G) = s$. Let $H = F - S$ where $F = K(n,n,n,n,n)$. Clearly, $t(e_i) \leqslant 3n$ for each $e_i \in S$. So,

$$t(H) \geqslant t(F) - 3ns,$$

with equality holds only if $t(e_i) = 3n$ for all $e_i \in S$. Since $t(H) = t(G) = t(F) - 3ns$, the equality above holds with $t(e_i) = 3n$ for all $e_i \in S$. Therefore each edge in S has an end-vertex in V_i and another end-vertex in $V_j (1 \leqslant i < j \leqslant 5)$. Moreover, S must induce a matching in F. Otherwise, equality does not hold or $\alpha'(H) > s$. By Lemma 2.8, we obtain

$$Q(G) = Q(F) - s(n-1)^2 + \binom{s}{2} + 3s\binom{n}{2},$$

whereas

$$\begin{aligned}Q(H) = {} & Q(F) - s(n-1)^2 + \binom{s}{2} - s_{12}(s_{13}+s_{14}+s_{15}+s_{23}+s_{24}+s_{25}) \\ & - s_{13}(s_{14}+s_{15}+s_{23}+s_{34}+s_{35}) - s_{14}(s_{15}+s_{24}+s_{34}+s_{45}) \\ & - s_{15}(s_{25}+s_{35}+s_{45}) - s_{23}(s_{24}+s_{25}+s_{34}+s_{35}) - s_{24}(s_{25}+s_{34}+s_{45}) \\ & - s_{25}(s_{35}+s_{45}) - s_{34}(s_{35}+s_{45}) - s_{35}s_{45} + 3s\binom{n}{2} \\ \leqslant {} & Q(G)\end{aligned}$$

and the equality holds if and only if $s = s_{ij}$ for $1 \leqslant i < j \leqslant 5$, or $s = s_{ij} + s_{kl}$ for $1 \leqslant i < j \leqslant 5, 1 \leqslant k < l \leqslant 5$, $\{i,j\} \cap \{k,l\} = \emptyset$. Moreover, $K(G) = K(F) - 3sn^2$ whereas

$$\begin{aligned}K(H) = {} & K(F) - 3sn^2 + s_{12}(s_{34}+s_{35}+s_{45}) + s_{13}(s_{24}+s_{25}+s_{45}) \\ & + s_{14}(s_{23}+s_{25}+s_{35}) + s_{15}(s_{23}+s_{24}+s_{34}) + s_{23}s_{45} + s_{24}s_{35} + s_{25}s_{34} \\ \geqslant {} & K(G)\end{aligned}$$

and the equality holds if and only if $s = s_{ij}$ for $1 \leqslant i < j \leqslant 5$. Hence,

$$Q(H) - 2K(H) \leqslant Q(G) - 2K(G)$$

and the equality holds if and only if $s = s_{ij}$ for $1 \leqslant i < j \leqslant 5$. Consequently, $\langle S \rangle = sK_2$ with $H \cong G$.

Suppose $H \sim G = K_{1,2}^{-sK_2}(n-1,n,n,n,n+1)$ for $n \geqslant s+3$. By Theorem 4.1 and Lemma 2.1, $H \in \mathscr{K}^{-s}(n-1,n,n,n,n+1)$ and $\alpha'(H) = \alpha'(G) = s$. Let $H = F - S$ where $F = K(n-1,n,n,n,n+1)$. Clearly, $t(e_i) \leqslant 3n+1$ for each $e_i \in S$. So,

$$t(H) \geqslant t(F) - s(3n+1),$$

with equality holds only if $t(e_i) = 3n+1$ for all $e_i \in S$. Since $t(H) = t(G) = t(F) - s(3n+1)$, the equality above holds with $t(e_i) = 3n+1$ for all $e_i \in S$. Therefore each edge in S has an end-vertex in V_1 and another end-vertex in $V_j (2 \leqslant j \leqslant 4)$. Moreover, S must induce a matching in F. Otherwise, equality does not hold or $\alpha'(H) > s$. By Lemma 2.8, we obtain

$$Q(G) = Q(F) - s(n-2)(n-1) + \binom{s}{2} + s\left(2\binom{n}{2} + \binom{n+1}{2}\right),$$

whereas

$$\begin{aligned}Q(H) = {} & Q(F) - s(n-2)(n-1) + \binom{s}{2} \\ & - (s_{12}s_{13} + s_{12}s_{14} + s_{13}s_{14}) + s\left(2\binom{n}{2} + \binom{n+1}{2}\right) \\ \leqslant {} & Q(G),\end{aligned}$$

and the equality holds if and only if $s = s_{1j} (2 \leqslant j \leqslant 4)$. Moreover, $K(G) = K(H) = K(F) - s(3n^2 + 2n)$. Hence, $2K(G) - Q(G) = 2K(H) - Q(H)$ if and only if $\langle S \rangle \cong sK_2$ with $H \cong G$.

Thus the proof is complete. □

Remark: Our results generalized Theorems 3 and 4 in (Zhao, 2005).

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